Subexponential Algorithms in Geometric Graphs via the Subquadratic Grid Minor Property : The Role of Local Radius

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Joint work with

Marin Bougeret, Daniel Gonçalves, Jean-Florent Raymond

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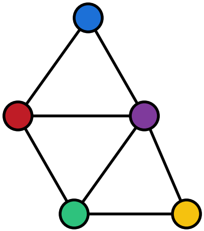
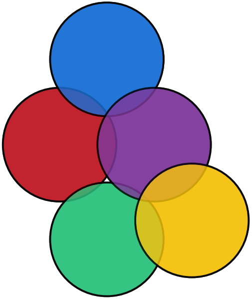
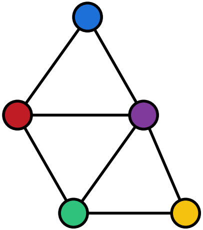
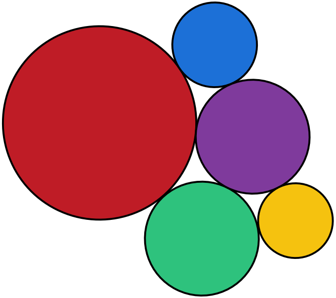
Geometric graphs : Unit disks



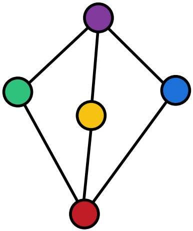


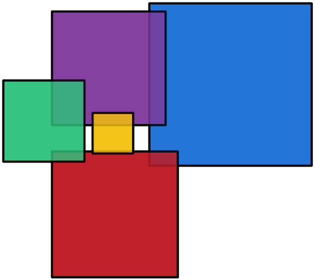
Geometric graphs : Disks in contact = planar graphs



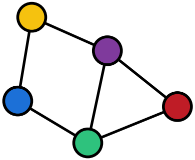


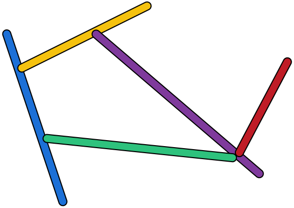
Geometric graphs : Axis parallel squares





Geometric graphs : Segments in contact





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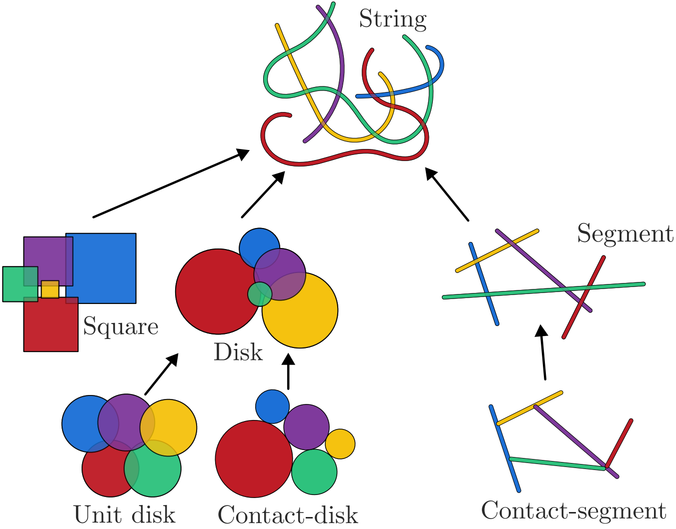
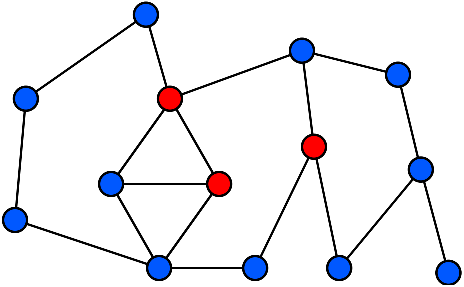


The Feedback Vertex Set problem (FVS)

The Feedback vertex set problem

Can we remove at most k vertices of G such that the resulting graph has no cycle ?



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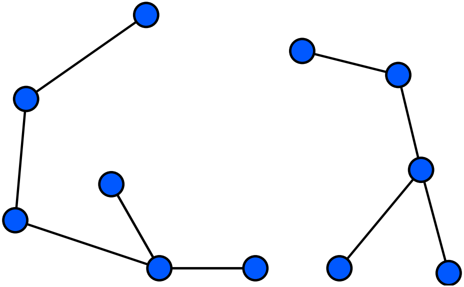

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Theorem (Li and Nederlof 2022)

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What about geometric graph classes ?

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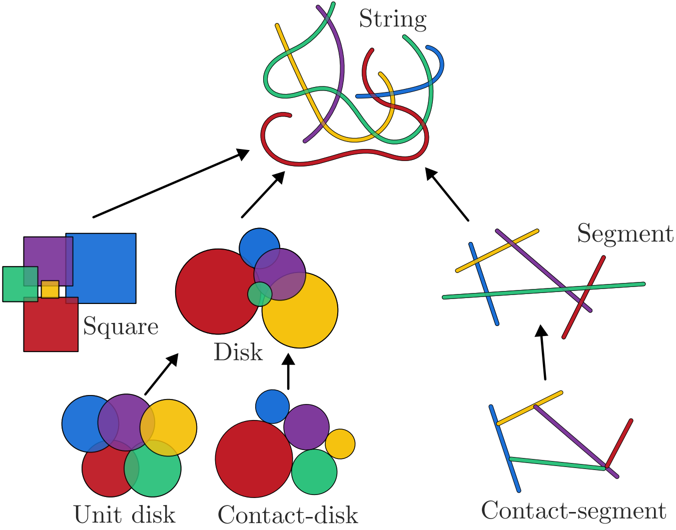

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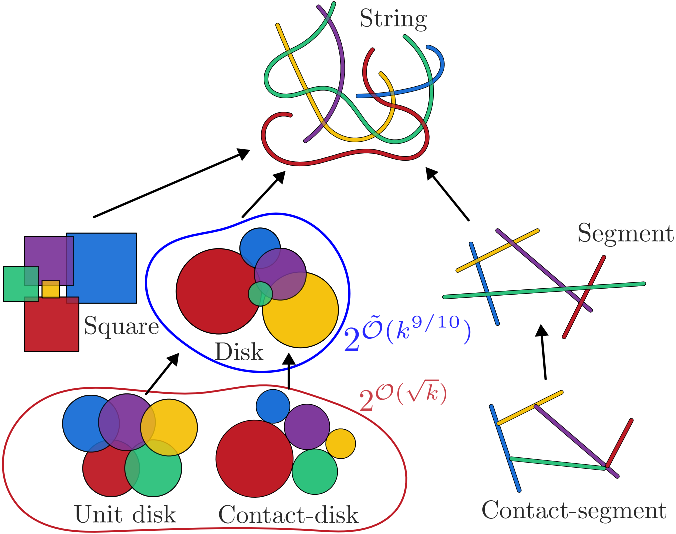

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Example of application

Theorem (Demaine et al. 2005)

FVS in planar graphs can be solved in time $2^{\mathcal{O}(\sqrt{k})} n^{\mathcal{O}(1)}$.

Bidimensionality

FVS is a bidimensional problem

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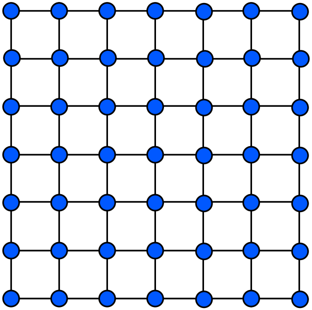

Figure: The 7×7-grid.

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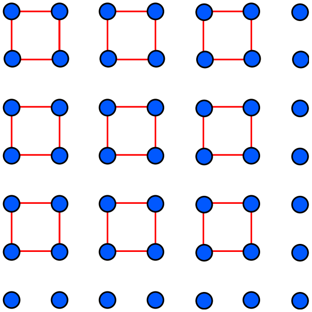

Figure: The 7×7-grid.

The SQGM property

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A graph class $\mathcal{C}$ has the SQGM property if there exists $c < 2$ such that for $G \in \mathcal{C}$, $\text{tw}(G) = \mathcal{O}(\boxplus(G)^c)$.

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Example: Grid Minor Theorem for planar graphs.

For G planar, we have $\mathrm{tw}(G) = \mathcal{O}(\boxplus(G))$.

Solving FVS in classes with the SQGM property

General method

We have $\mathrm{tw}(G) = \mathcal{O}(\boxplus(G)^c)$, with $c < 2$.

Two cases :

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- Otherwise :
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$$\Rightarrow \boxplus(G) > \sqrt{k},$$
$\Rightarrow$ G does not contain a FVS of size k.

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Relation between the grid number and the treewidth

$\boxplus(G)$: Biggest t such that G contains the $t \times t$-grid as a minor.

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The case of the cliques

$\mathrm{tw}(K_n) = n - 1$ and $\boxplus(K_n) \leq \sqrt{n}$,

so $\mathrm{tw}(K_n)$ **is not subquadratic in** $\boxplus(K_n)$.

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Any geometric graphs class containing the cliques **does not have** the SQGM property.

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ASQGM (Almost-SQGM)

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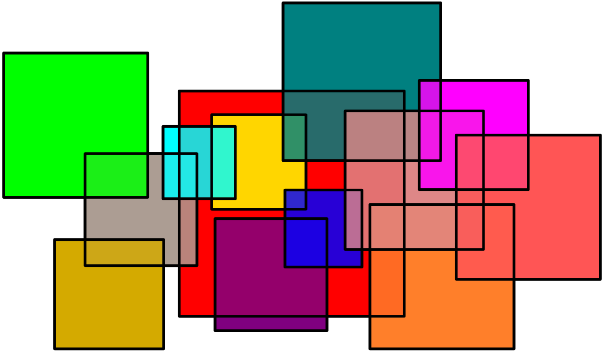
$$\text{Unit-disks : } \mathrm{tw}(G) = \mathcal{O}(\omega(G)\boxplus(G)).$$

$$\text{Intersecting squares : } \mathrm{tw}(G) \stackrel{?}{=} \mathcal{O}(f(G)\boxplus(G)).$$

$$\text{Contact-segments : } \mathrm{tw}(G) \stackrel{?}{=} \mathcal{O}(f(G)\boxplus(G)).$$

How to find an ASQGM relation

Local arrangement graph definitition $G_{\mathcal{S}}(v)$



A representation $\mathcal{S}$

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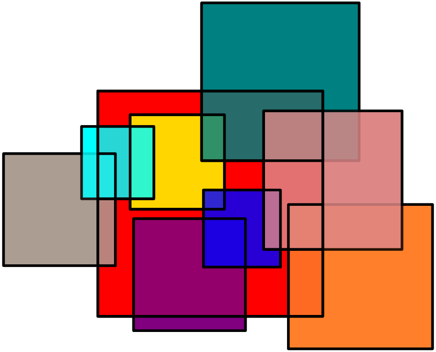

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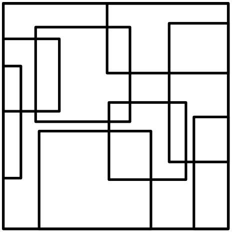

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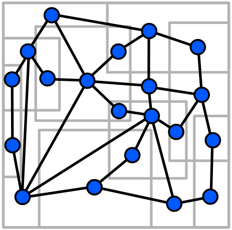

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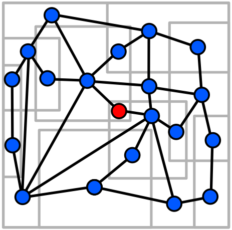

$$radius(G_{\mathcal{S}}(v)) = 4.$$

How to find an ASQGM relation

Local radius (Lokshtanov et al. 2023)

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Using ASQGM for subexponetial FVS in squares

Problem : Is there a FVS of size at most k in a square graph G ?

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- A bound $\mathrm{LR}(G) \leq k^{\epsilon'}$ would gives $\mathrm{tw}(G) = \mathcal{O}(k^{1/2+\epsilon+\epsilon'})$,
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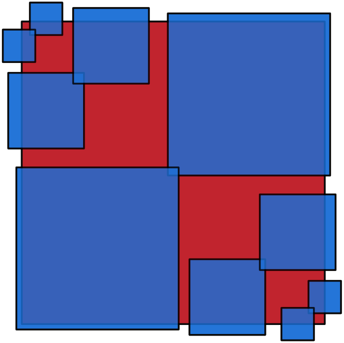
$\Rightarrow$ Taking ϵ and ϵ' small we obtain a **subexponential time algorithm** !

Toward an ASQGM relation for squares

The local radius in not bounded by $\omega(G)$.



We have no big cliques : $\omega(G) = 3$

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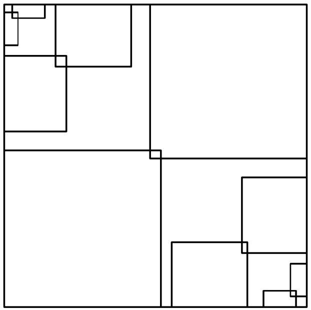

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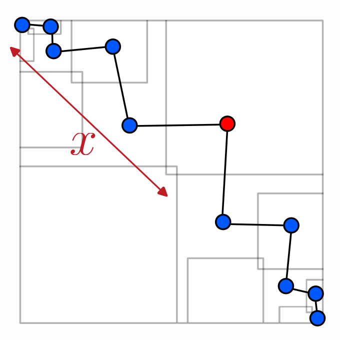

$\mathit{radius}(A[v]) = x$. Local radius as big as wanted !

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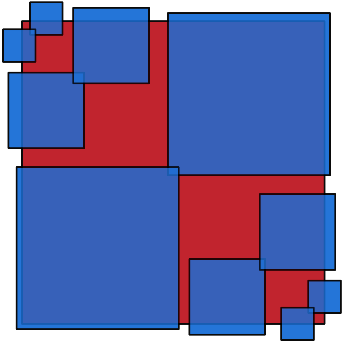

$radius(A[v]) = x$. Local radius as big as wanted !

But we have a **big matching.**

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Bound the local radius

We denote $\mu^*(v)$ the size of the biggest matching in $N(v)$.

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In geometric graph of squares,

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Theorem (Preproccessing by Lokshtanov et al. + counting argument)

We can reduce in subexponential time to $2^{\tilde{\mathcal{O}}(k^{1-\epsilon})}$ instances with

$$\mu^*(G) \leq k^{3\epsilon}.$$

The ASQGM method for the intersections of squares

Overall complexity

$$\mathrm{tw}(G) = \mathcal{O}(\underbrace{\omega(G)}_{\mathcal{O}(k^{\epsilon})} \cdot \mathrm{LR}(G) \cdot \underbrace{\boxplus(G)}_{\mathcal{O}(\sqrt{k})})$$

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Taking $\epsilon = 1/10$, we obtain :

Theorem (This work)

The FVS problem for geometric graphs of squares can be solved in time

$$2^{\tilde{\mathcal{O}}(k^{9/10})} n^{\mathcal{O}(1)}.$$

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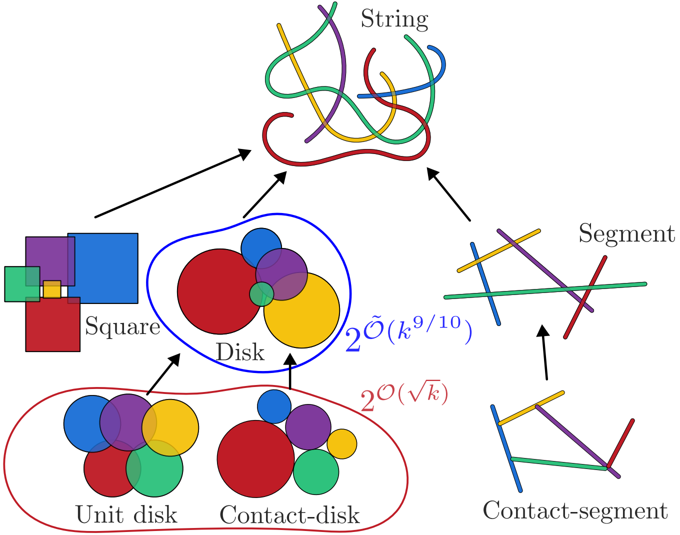

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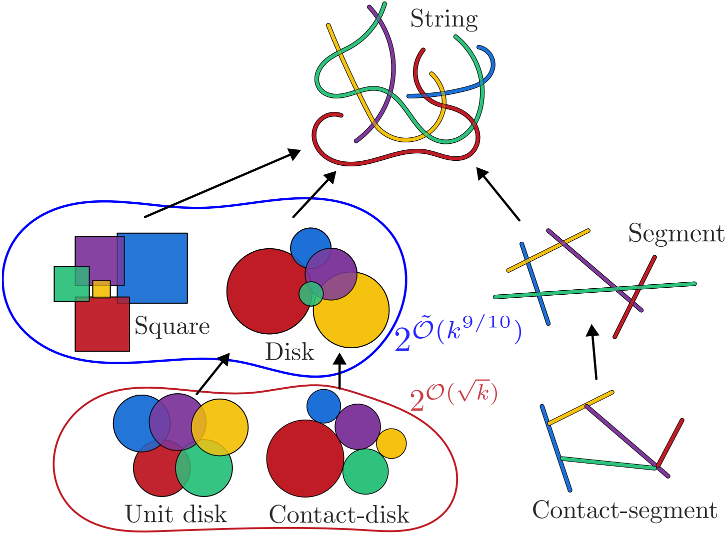

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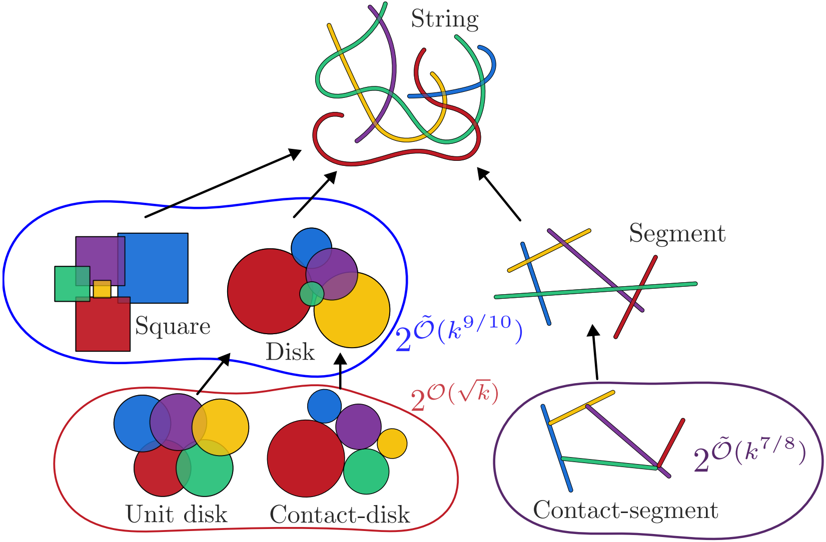

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Conclusion

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Take-away message : Bounding the local radius is the key !

Conclusion

Our results :

- $2^{o(k)}n^{\mathcal{O}(1)}$ algorithm for FVS in square graphs.

In the full version on arXiv :

- Similar result for contact-segments.
- ETH-based lower bounds.

Take-away message : Bounding the local radius is the key !

Thanks for your attention !

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