Determining a graph from its reconfiguration graph

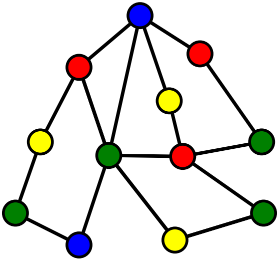
Gaétan Berthe
LIMOS, Université Clermont Auvergne

Joint work with
Caroline Brosse, **Brian Hearn**, **Jan van den Heuvel**,
Pierre Hoppenot, **Théo Pierron**

Work initiated as a research group of JGA 2023

Slides based on the talk of Brian Hearn at Eurocomb'25

Proper k-coloring



Definition

A k-coloring of a graph G is an assignment of **at most** k colors to the vertices of G, where no two adjacent vertices are assigned the same color.

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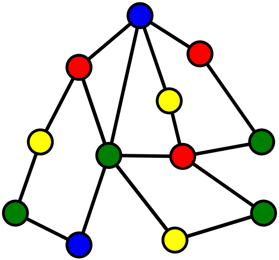

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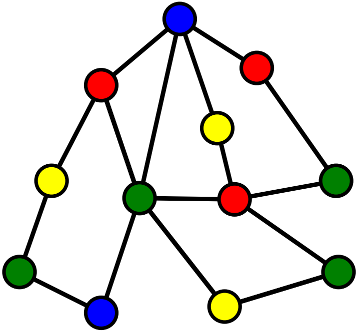
Definition

The chromatic number $\chi(G)$ of a graph G is the smallest k such that the graph G admits a proper k-coloring.

Recoloring a graph



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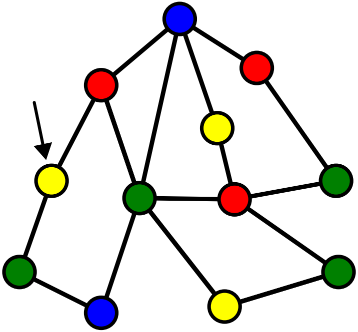

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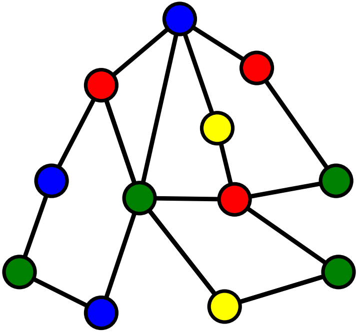

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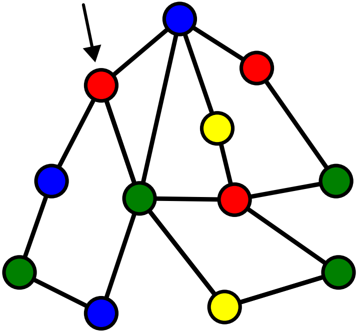

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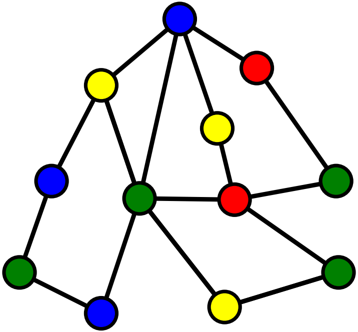

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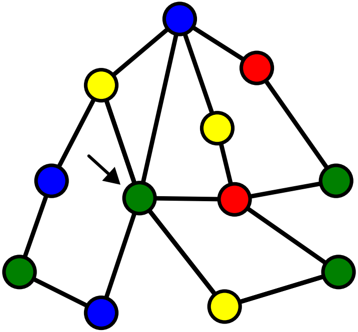
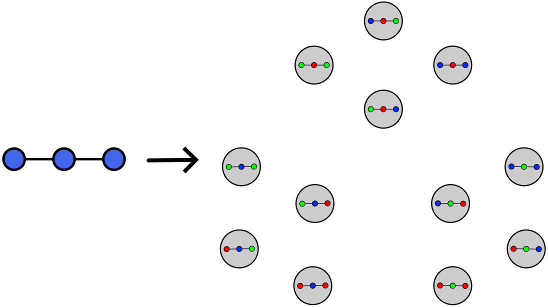


A frozen vertex

The k-recoloring graph $\mathcal{C}_k(G)$

G

The k-recoloring graph $\mathcal{C}_k(G)$



G k-colorings of G

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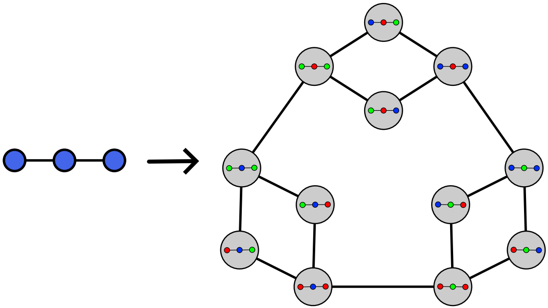

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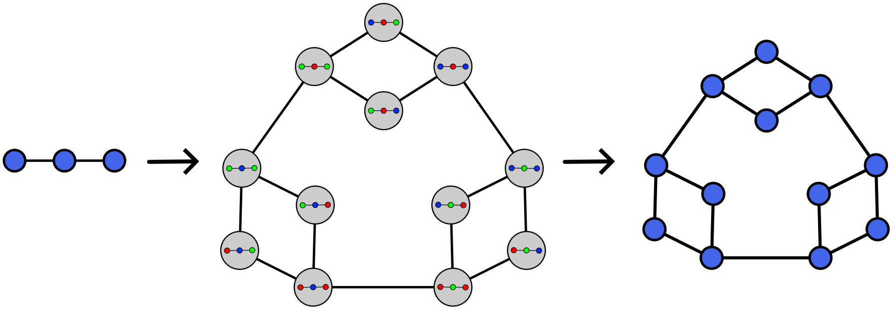

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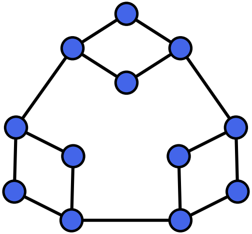
The k-recoloring graph $\mathcal{C}_k(G)$

$$G \xleftarrow{\qquad ? \qquad}$$



$\mathcal{C}_k(G)$

Determining the graph G from $\mathcal{C}_k(G)$

Question

Given the recoloring graph $\mathcal{C}_k(G)$, can we reconstruct G?

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Observation

If $k < \chi(G)$, the graph $\mathcal{C}_k(G)$ has no vertices.

Determining the graph G from $\mathcal{C}_k(G)$

Conjecture (Asgarli, Krehbiel, Levinson, Russell (2024))

A graph G is uniquely determined by the family $\{\mathcal{C}_k(G)\}_{k\in\mathbb{N}}$.

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Our result:

Theorem (B., Brosse, Han, van den Heuvel, Hoppenot, Pierron (2025))

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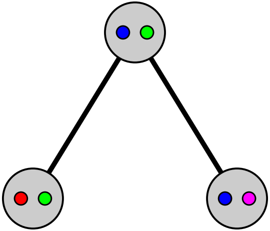
The cliques of $\mathcal{C}_k(G)$



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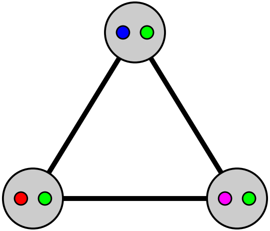

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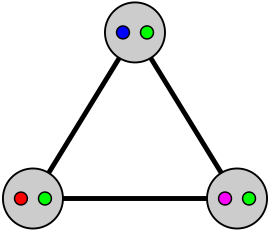

Observation

The colorings in a triangle of $\mathcal{C}_k(G)$ differ on a single vertex.

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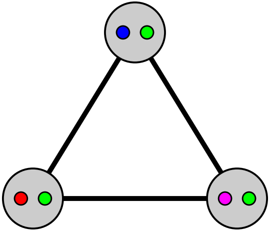
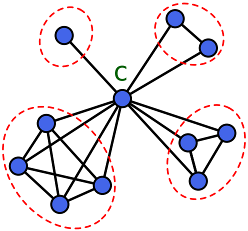


Observation

The colorings in a **clique** of $\mathcal{C}_k(G)$ differ on a single vertex.

Determining $|V(G)|$ from $\mathcal{C}_k(G)$



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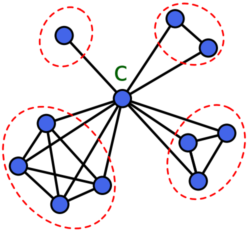

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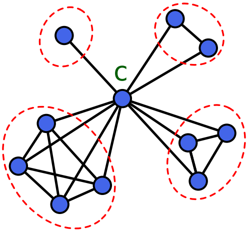
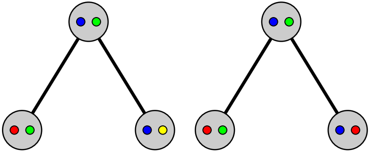


Observation

If a coloring c uses strictly less than k colors, the neighborhood of c in $\mathcal{C}_k(G)$ is $|V(G)|$ disjoint cliques.

Taking $k > \chi(G)$ assures that $n = |V(G)|$ can be determined from the recoloring graph $\mathcal{C}_k(G)$.

Non-edge



Determining $E(G)$ from $\mathcal{C}_k(G)$

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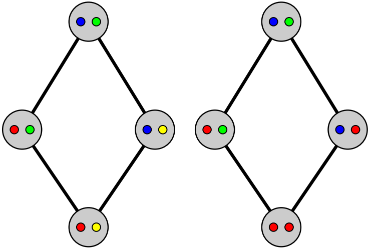

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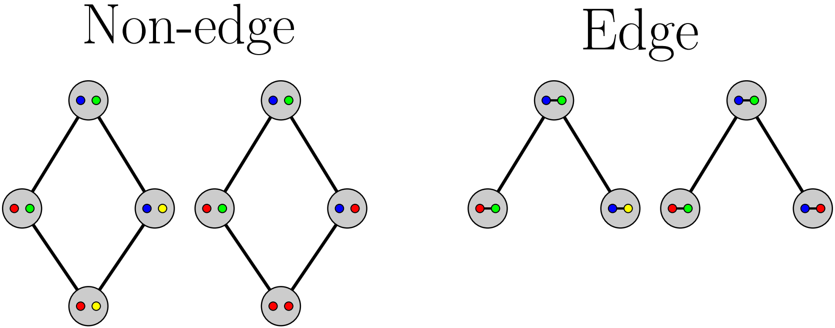

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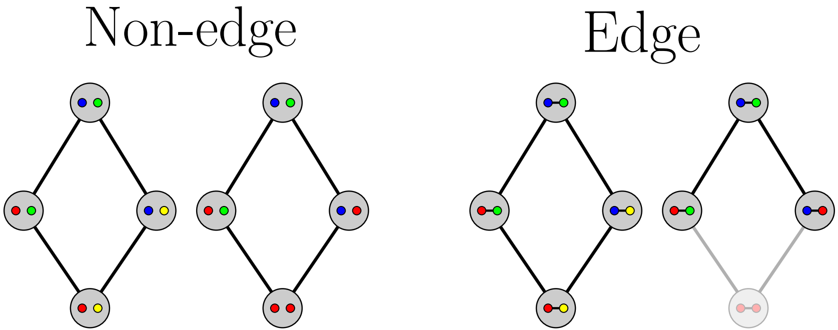

Reconstructing G

For each vertex c of $\mathcal{C}_k(G)$, we create a graph G_c with:

- The vertices of G_c are the maximal cliques on the neighbors of c,
- and the edges are decided by looking at the C_4 containing c.

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Observation 2

For a k-coloring c using less than k colors, we have $G_c = G$.

$\Rightarrow$ **If $k > \chi(G)$, the largest of the constructed G_c graphs is G.**

The case $k = \chi(G)$

- For $k < \chi(G)$, the graph G cannot be determined from $\mathcal{C}_k(G)$,
- for $k > \chi(G)$, the graph G can be determined from $\mathcal{C}_k(G)$.

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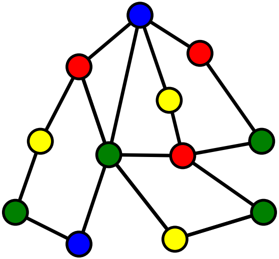
Observation

It is not always possible to determine G from $\mathcal{C}_k(G)$ when $k = \chi(G)$.

Example: The set of connected bipartite graphs.

What about other types of reconfiguration?

- Kempe-chain reconfiguration.



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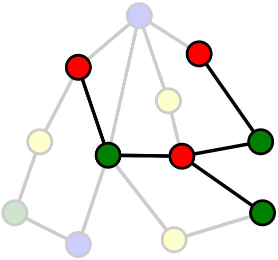

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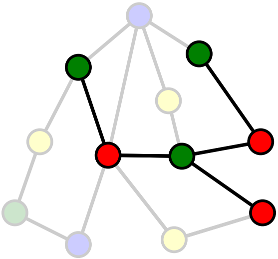

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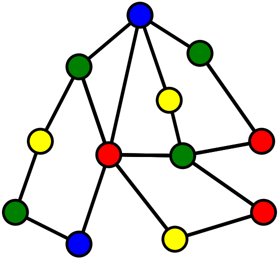

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- Independent-set reconfiguration (token sliding, token jumping).

Directions for further research

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Thanks for your attention !