Claquer les cliques

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Joint work with

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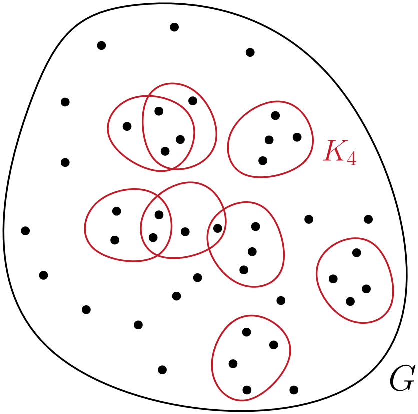
For $r \geq 2$ fixed :

The K_r-Hitting problem

Input : A graph G and an integer k.
Output : Is there a set $S \subseteq V(G)$ of size at most k intersecting all K_r.



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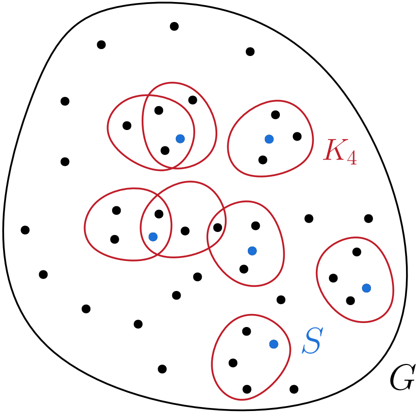

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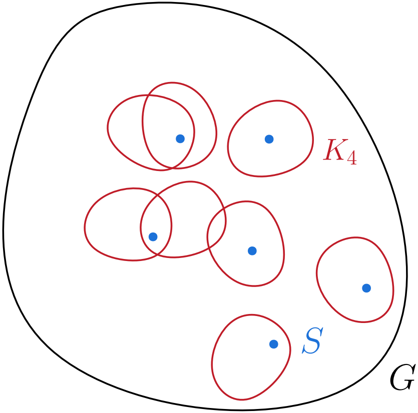

Known results for K_r-Hitting

Lower bound

K_r-Hitting cannot be solved in time $2^{o(k)} n^{O(1)}$ (under ETH).

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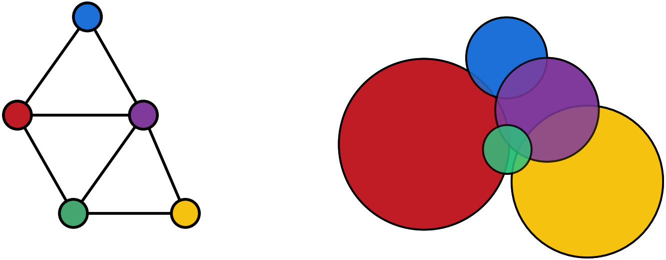
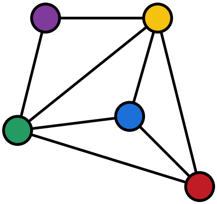
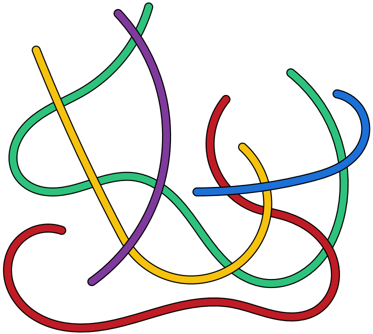
What about dense graph classes ?

Intersection graphs : Disk graphs

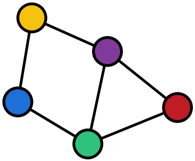


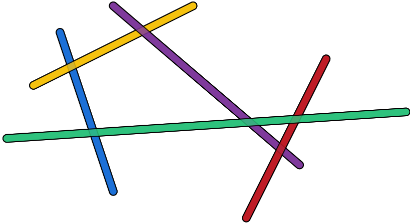
Intersection graphs : String graphs





Intersection graphs : Segment graphs





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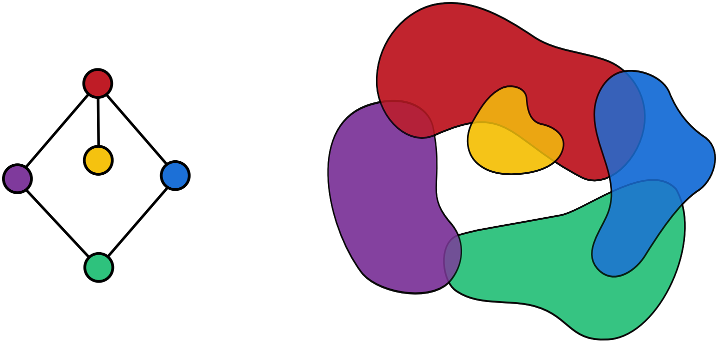

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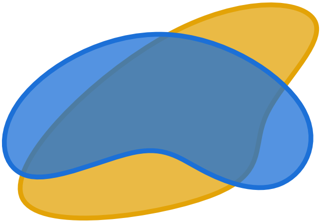
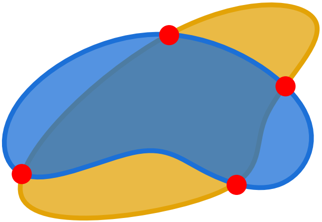
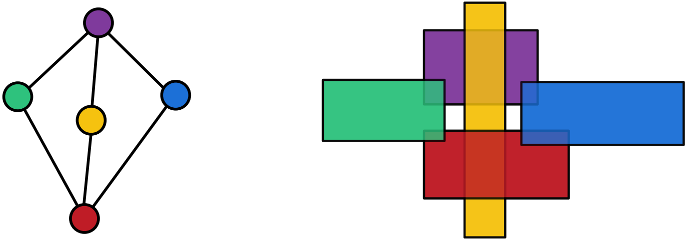
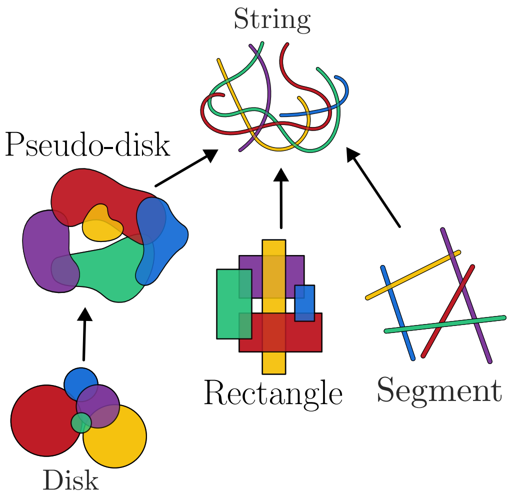




Intersection graphs : Rectangle graphs



Complexity of the TRIANGLE HITTING problem ($r = 3$)



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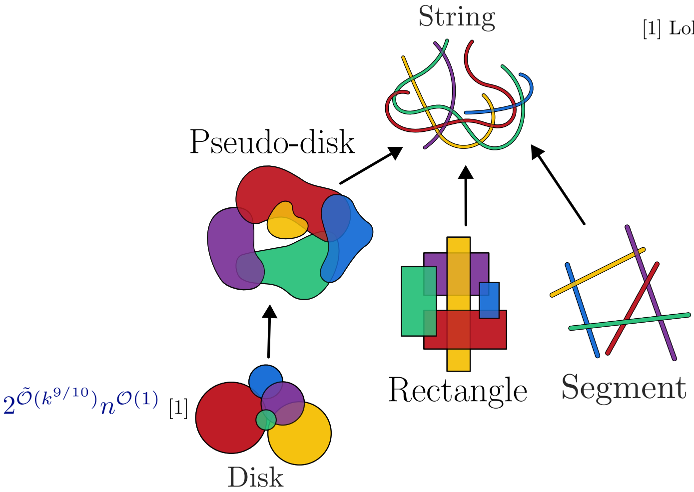

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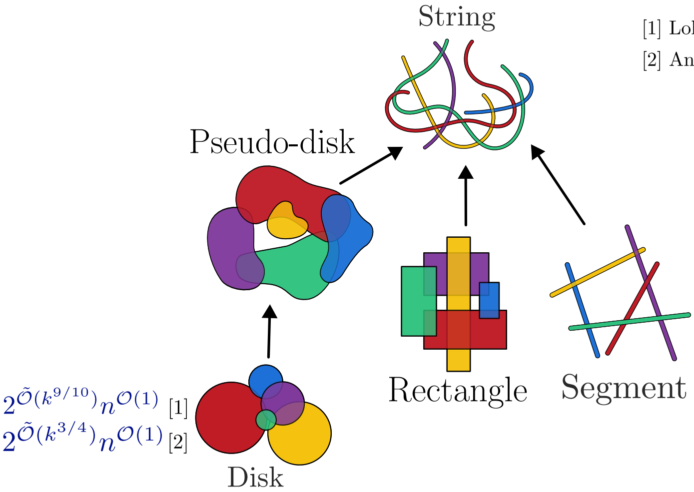

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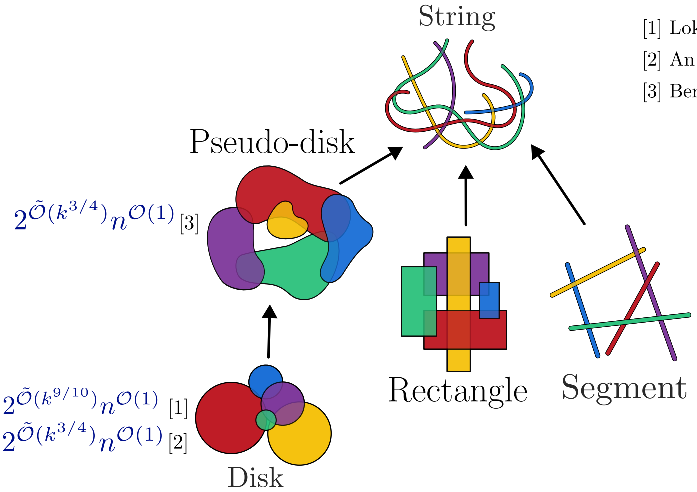

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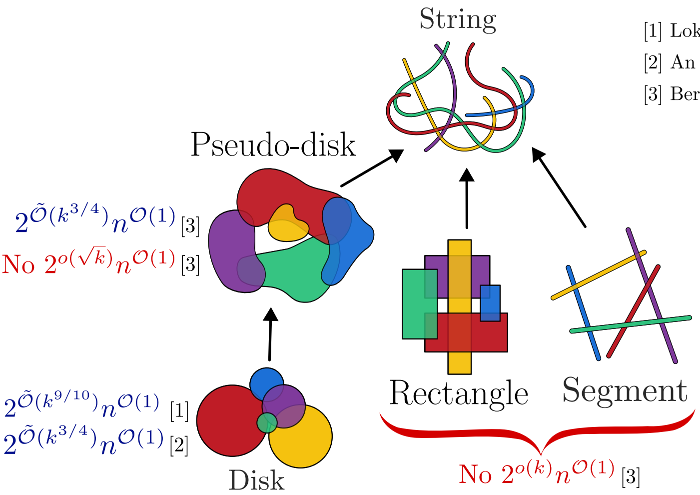

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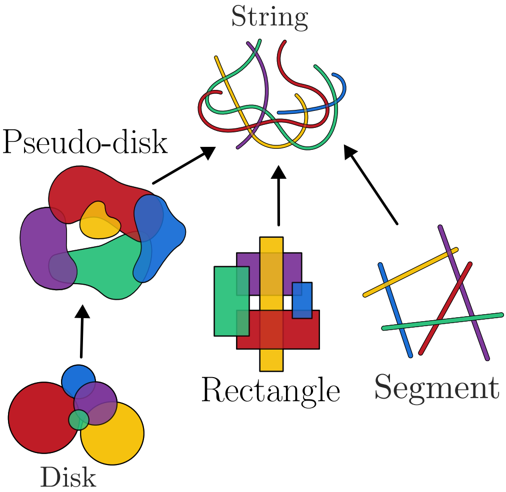
Our results for the K_r-HITTING problem



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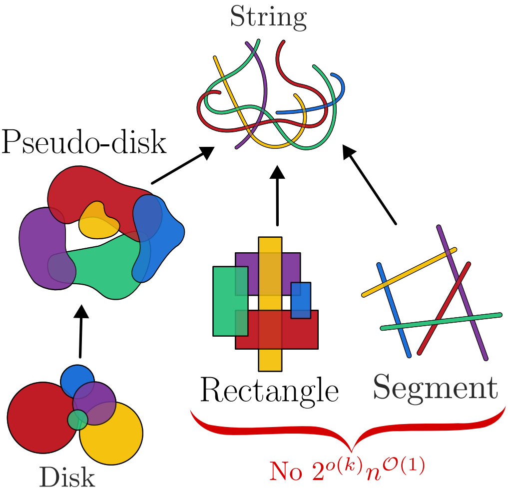

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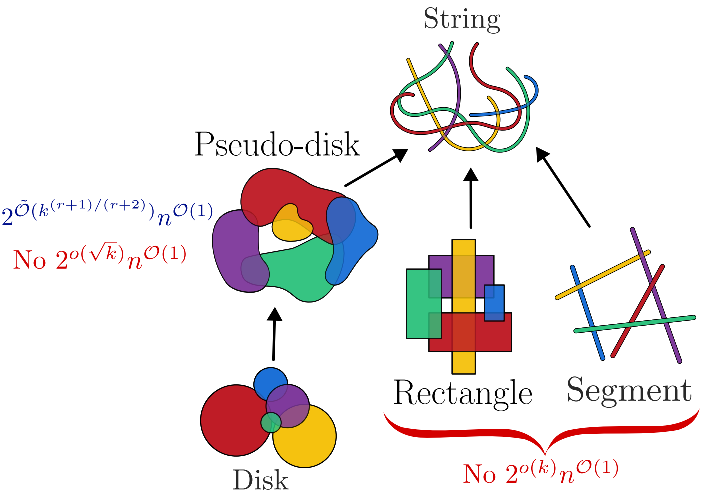

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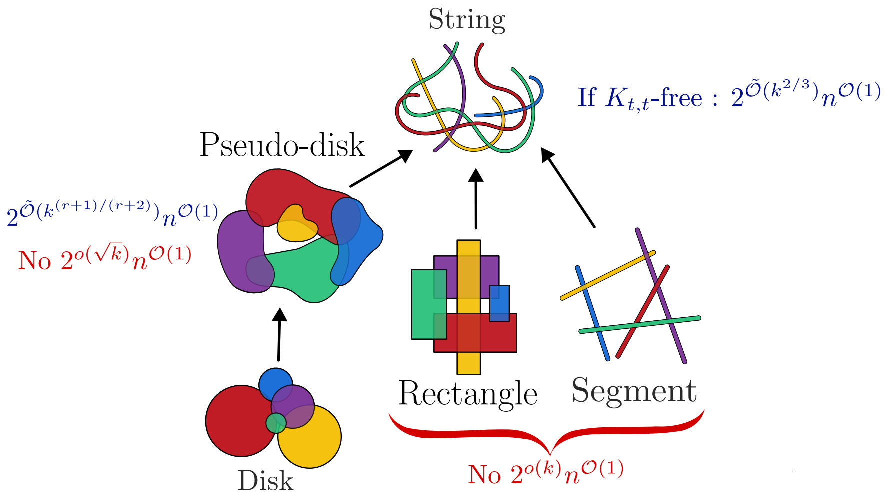

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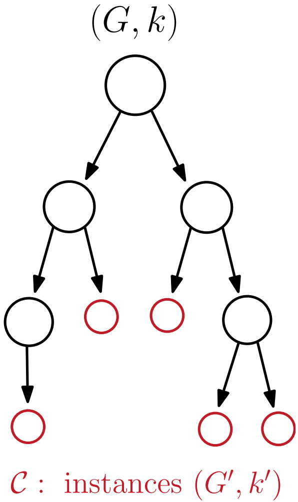

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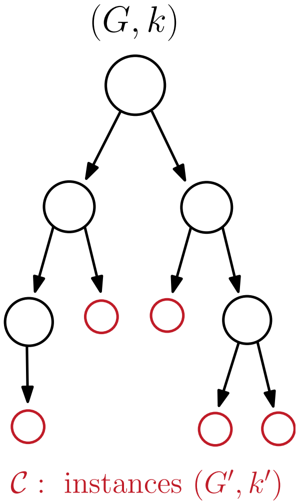

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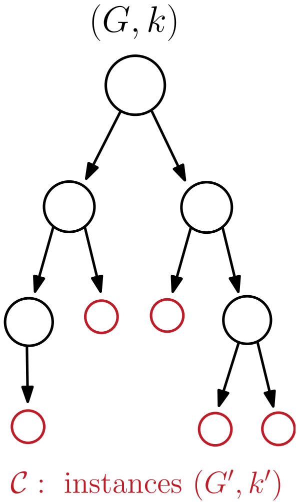

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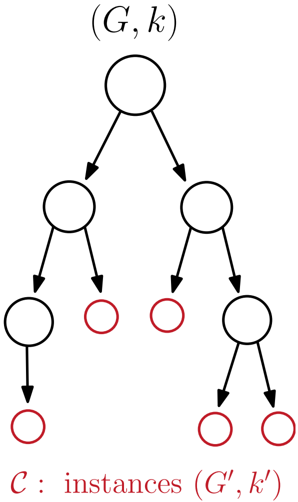

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$$\mathrm{tw}(G') = O(\sqrt{m}) = \sqrt{\omega \cdot n(G')} = o(k).$$

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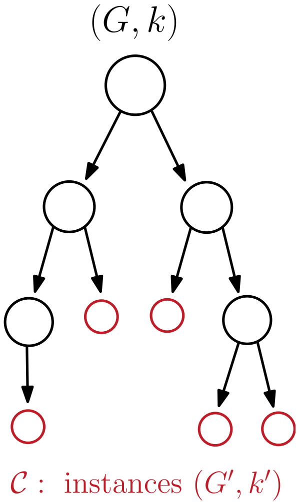

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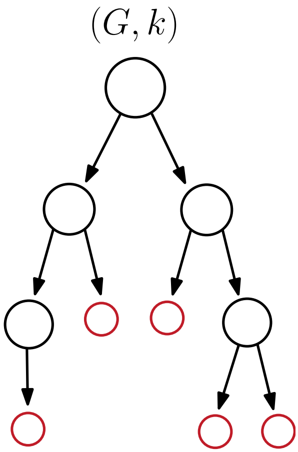

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So if $|\mathcal{C}| = 2^{o(k)}$, the running time is :

$$|\mathcal{C}| 2^{\text{tw}(G')} n^{O(1)} = 2^{o(k)} 2^{o(k)} n^{O(1)}$$
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Preprocessing step : Removing the big cliques

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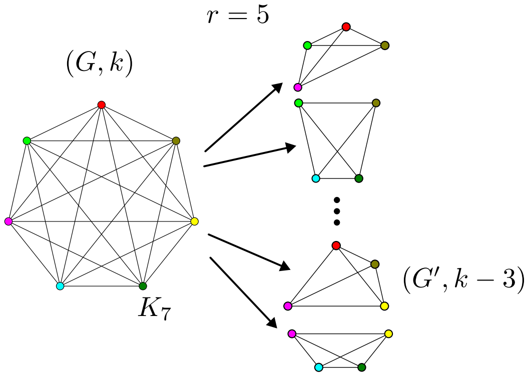
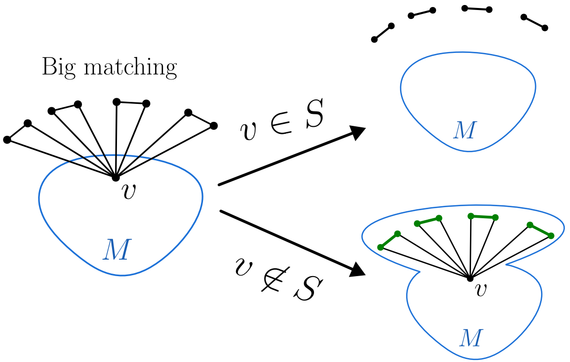
Folklore branching idea :



Result : $2^{O(k^{1-\epsilon})}$ instances (G', k') where G' is K_p-free.

Example for $r = 3$: big matching

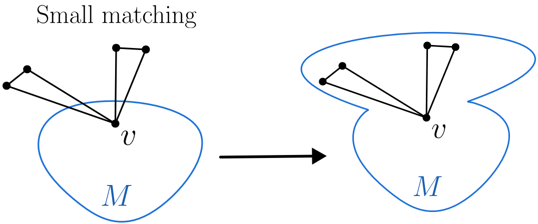
Step 0 : M a K_3-Hitting set with $|M| \leq 3k$.
Step 1 : For each vertex $v \in M$ with a big ($\geq k^\epsilon$) matching in $N(v) \setminus M$, we generate two instances :



If $k+1$ disjoint edges marked, we have a NO-instance.

Example for $r = 3$: Small matching

Step 2 : M' obtained by absorbing for each $v \in M$ the (small) maximum matching in its neighborhood outside of M.

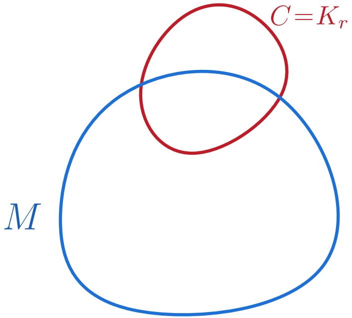


After those steps :

- $|M'| \leq |M| + 2k + k^{\epsilon} = O(k^{1+\epsilon})$,
- each triangle has at least two vertices in M'.

Generalization for $r \geq 3$

Definition : Given (G, M, k), a r-clique C of type i is an r-clique with $|C \cap M| = i$.



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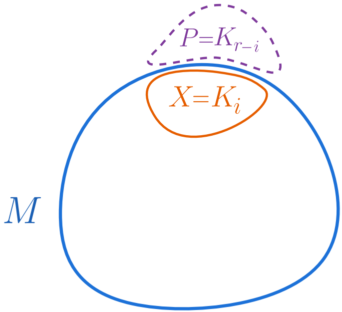

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Rethinking of the previous case $\mathbf{r = 3}$:

- **Initially :** No triangles of type 0 : M is a triangle-Hitting set.
- **After the virtual branching :** No more triangles of type 1.

Bounding the number of cliques

Problem : When $i \geq 2$, in general there can be a lot of such i-cliques.

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Example for **pseudo-disks** :

Theorem (Chiba et al. 1985)

Any graph G with n vertices and degeneracy d has $O(id^{i-1}n)$ i-cliques.

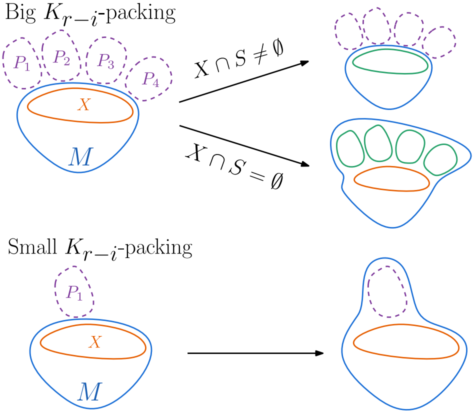
Theorem (Lokshtanov et al. 2024)

Pseudo-disk graphs have degeneracy $O(\omega(G))$.

In short : The number of cliques to consider in M is $O(\omega^r|M|)$.

Branching for X of type i

The same two cases :



An instance is now G, k, M and a family $\mathcal{D}$ of sets hit by the solution.
If D contains more than k disjoint sets, we have a NO-instance.

For general r : Given (G, k) , perform $r-1$ rounds as follows :

- Initially: M_0, no type 0 r-clique, $|M_0| = O(k)$.

Back to our contribution for K_r-HITTING

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- After round 1: M_1, no more r-clique type 1, $|M_1| = O(k^{1+\epsilon})$.
- ...
- After round $r-1$: M_{r-1}, no more r-clique of type $r-1$, $|M_{r-1}| = O(k^{1+r\epsilon})$.

And we are done, output M_{r-1} and $\mathcal{D}$.

Obtained results

For a fixed r and $\mu \geq 0$, let $\mathcal{G}$ be a hereditary graph class such that for all $G \in \mathcal{G}$:

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Examples of applications

The K_r-Hitting problem can be solved in time :

- $2^{\tilde{O}(k^{(r+1)/(r+2)})} n^{O(1)}$ for **Pseudo-disk graphs**.

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- $2^{\tilde{O}(k^{2/3})} n^{O(1)}$ for H-**minor-free graphs**.

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Thanks for your attention !