The Complexity of the L(a,b)-Edge-Labelling

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Joint work with

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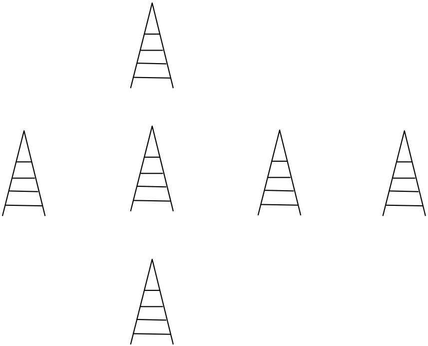
The $L(a, b)$-Labelling

Introduced as an antenna problem



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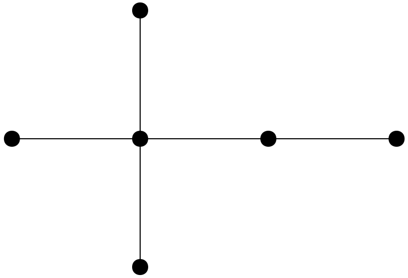

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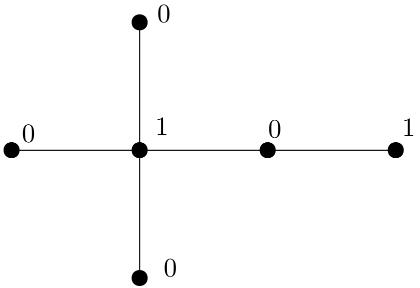

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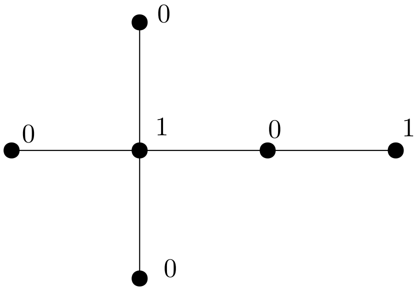
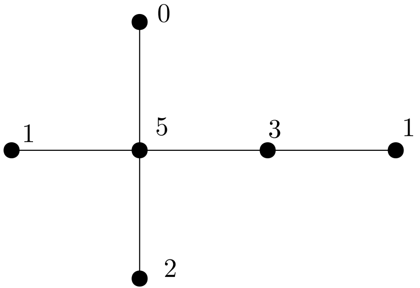


Problem : interferences for antennas at distance 2.

Introduced as an antenna problem



A safer frequency assignment : a $L(2, 1)$-Labelling.

Definitions

$L(a, b)$-LABELLING

Let $a, b \geq 0$ and $G = (V, E)$ a graph,
a $L(a, b)$-LABELLING is a function:

$$c : V \rightarrow \mathbb{N}$$

such that for every $v, v' \in V$:

$$dist(v, v') = 1 \Rightarrow |c(v) - c(v')| \geq a.$$
$$dist(v, v') = 2 \Rightarrow |c(v) - c(v')| \geq b.$$

Definitions

$L(a, b)$-k-Labelling

A $L(a, b)$-k-Labelling is a $L(a, b)$-Labelling which only uses the labels $\{0, ..., k-1\}$.

$L(a, b)$-k-Edge-Labelling

The previous definitions can be adapted to edge labelling.

Examples

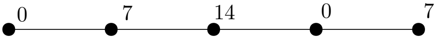


Figure: a $L(7, 5)$-15-LABELLING.

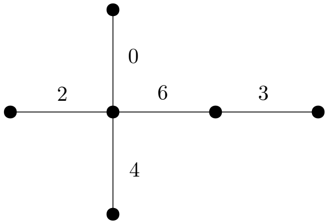


Figure: a $L(2, 1)$-7-EDGE-LABELLING.

Two useful properties of $L(a, b)$-LABELLING

For $c : V \to \{0, ..., k-1\}$ a $L(a, b)$-k-LABELLING:

Symmetry

$\overline{c}$ defined as $\overline{c}(v) = k - 1 - c(v)$ is still a $L(a, b)$-k-LABELLING.

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Linearity

G have a $L(a, b)$-k-Labelling $\iff$ G have a $L(\lambda a, \lambda b)$-λk-Labelling.

The $L(a, b)$-k-LABELLING problem

$L(a, b)$-k-LABELLING problem

Parameters : integers a, b and k

Input : a graph $G = (V, E)$.

Question : is there an $L(a, b)$-LABELLING of G using the labels $\{0, ..., k - 1\}$?

$L(a, b)$-Edge-Labelling complexity

Previous works : the vertex-labelling case

Lots of different results depending on the graph class.

Examples with trees :

Theorem (Fiala J., Kloks T., Kratochv íl, J. 2001)

The $L(a,1)$-LABELLING problem on trees is polynomial.

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Examples with trees :

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The $L(a, 1)$-LABELLING problem on trees is polynomial.

Theorem (Fiala J., Golovach P.A., and Kratochv íl, J. 2008)

If $a \nmid b$, the $L(a, b)$-LABELLING problem on trees is NP-COMPLETE.

Previous works : the edge-labelling case

Theorem (Mahdian M. 2002)

The $L(1,1)$-k-EDGE-LABELLING problem is polynomial for $k < 4$ and NP-COMPLETE for $k \geq 4$.

Theorem (Knop D., Masarík T. 2018)

The $L(2,1)$-k-EDGE-LABELLING problem is polynomial for $k < 6$ and NP-COMPLETE for $k \geq 6$.

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The $L(2, 1)$-k-EDGE-LABELLING problem is polynomial for $k < 6$ and NP-COMPLETE for $k \geq 6$.

And for other values of a, b ?

Our result

Theorem (B. G., Barnaby M., Paulusma D., Smith S. WALCOM 2022)

For all $a, b \geq 0$, except $a = b = 0$, there exists k so that the $L(a, b)$-k-Edge-Labelling problem is NP-complete.

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Proof

The proof is done by reduction from known NP-complete problems, using gadgets.

We have different cases depending on the value of $\frac{b}{a}$.

Gadget

Our reductions uses gadgets to encode a SAT variation problem with graphs :

- Variable gadget: represents the variable, the labelling encodes the value.
- Clause gadget: represents the clause, the labelling encodes the SAT problem.

Each clause gadget is connected to the associated variable gadgets, to make them interfere.

Example

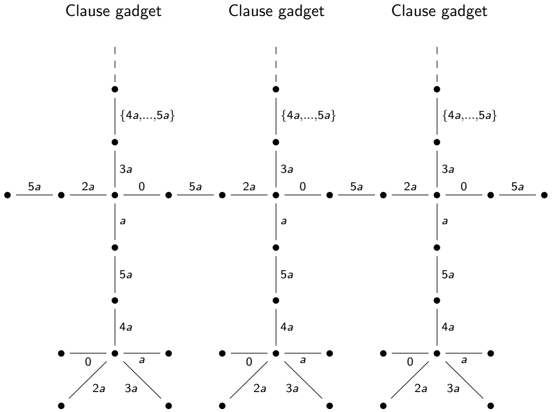

Figure: Variable gadget for $1 < \frac{b}{a} \leq 2$, $k = 5a + 1$, reduction from NAE-3-SAT.

Reductions

Regime	reduction from	k
$a = 0$ and $b > 0$	3-COL	$3b$
$2 < \frac{b}{a}$	NAE-3-SAT	$O(b)$
$1 < \frac{b}{a} \leq 2$	NAE-3-SAT	$5a + 1$
$\frac{b}{a} = 1$	3-COL (previous work)	$4a$
$\frac{2}{3} < \frac{b}{a} \leq 1$	3-COL	$3a + b + 1$
$\frac{b}{a} = \frac{2}{3}$	1-in-3-SAT	$4a$
$\frac{1}{2} < \frac{b}{a} < \frac{2}{3}$	2-in-4-SAT	$a + 4b + 1$
$0 < \frac{b}{a} \leq \frac{1}{2}$	NAE-3-SAT (previous work)	$3a + 1$
$a > 0$ and $b = 0$	3-COL (trivial)	$3a + 1$

A helpful tool

Creation of a Python program able to:

- List possibles $L(a, b)$-k-Edge-Labelling of a graph G.
- Remove the symmetric labellings.
- Add constraints on the possible labelling of some edges.

Example

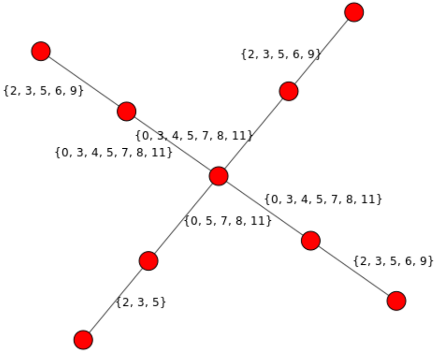

Figure: $L(3, 2)$-12-EDGE-LABELLING of the extended 4-star. The bottom left leaf is the chosen edge for removing the symmetric labelling.

Conclusion

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What was done

For $a, b \geq 0$, except $a = b = 0$, found a k such that the $L(a, b)$-k-EDGE-LABELLING problem is NP-COMPLETE.

What still need to be done

Describe the set of values k such that the $L(a, b)$-k-EDGE-LABELLING problem is NP-COMPLETE.

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Thanks for your attention!