Kick the cliques

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Joint work with

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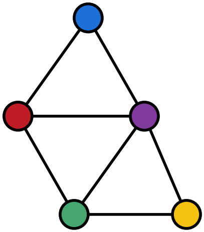
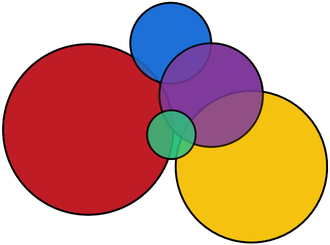
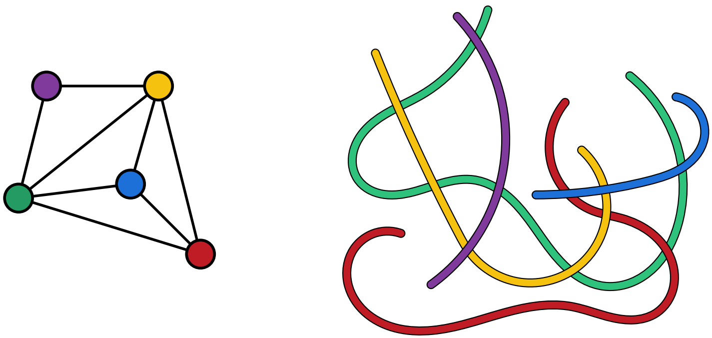
Geometric graphs : Disk graphs

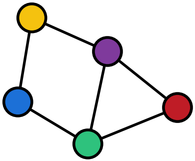


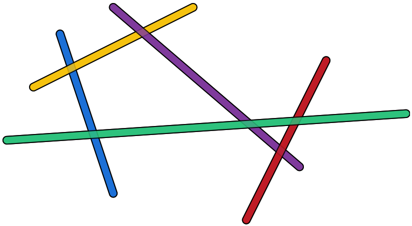


Geometric graphs : String graphs

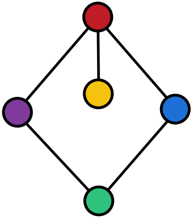


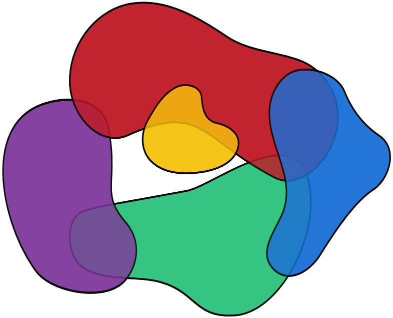
Geometric graphs : Segment graphs





Geometric graphs : Pseudo-disk graphs





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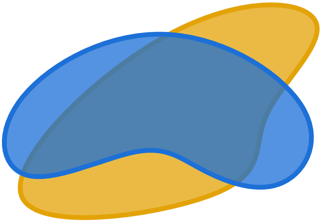

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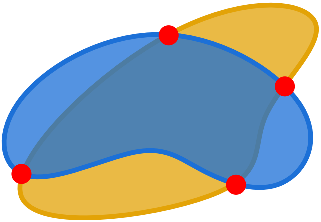
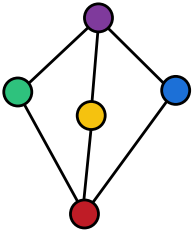
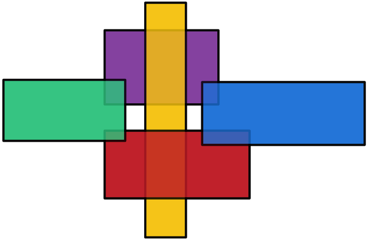
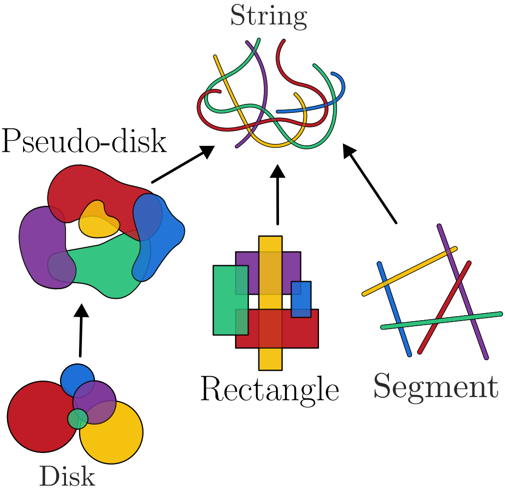


Geometric graphs : Rectangle graphs



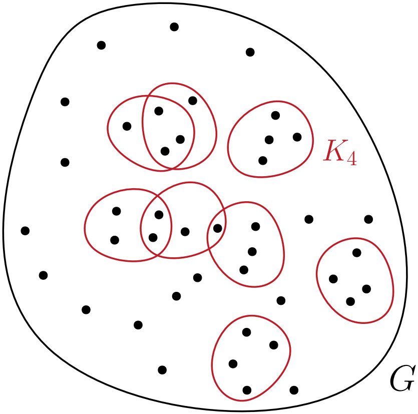



String
Pseudo-disk
Rectangle
Segment
Disk

For $r \geq 2$ fixed :

The K_r-HITTING problem

Input : A graph G and an integer k.
Output : Is there a set $S \subseteq V(G)$ of size at most k intersecting all K_r.



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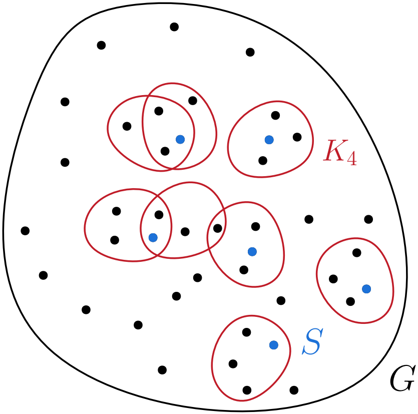

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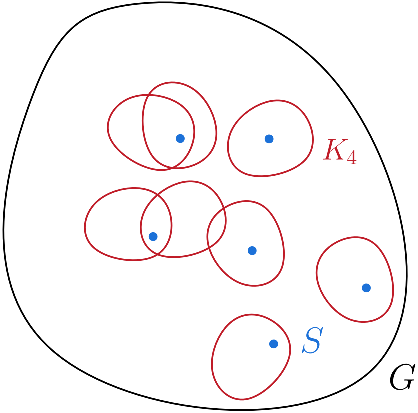

Known results about the K_r-HITTING problem

Approximation (Dvořák et al. 2023)

The K_r-HITTING problem admit an EPTAS for pseudo-disk graphs (still true for $\mathcal{F}$-HITTING with $\mathcal{F}$ a finite family of connected graphs).

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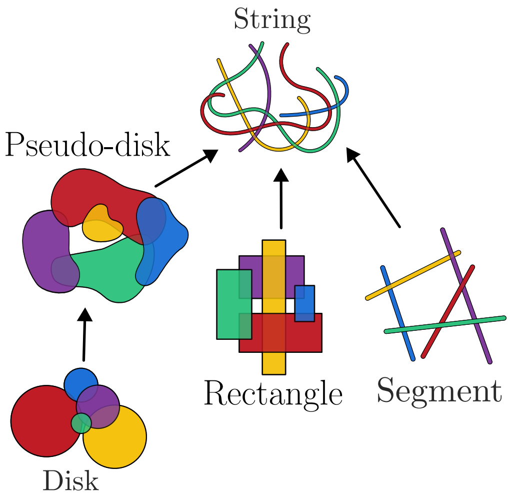
What about parameterized complexity ?

Complexity of the TRIANGLE HITTING problem ($r = 3$)



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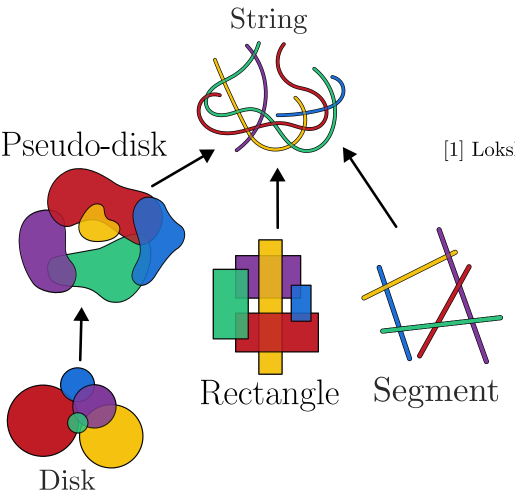

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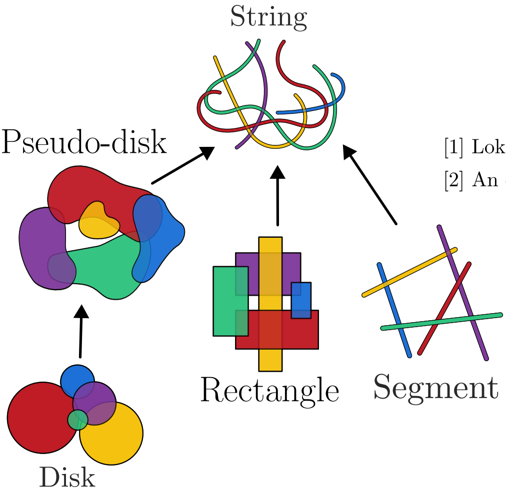

$2^{\tilde{\mathcal{O}}(k^{9/10})} n^{\mathcal{O}(1)}$ [1]
$2^{\tilde{\mathcal{O}}(k^{3/4})} n^{\mathcal{O}(1)}$ [2]

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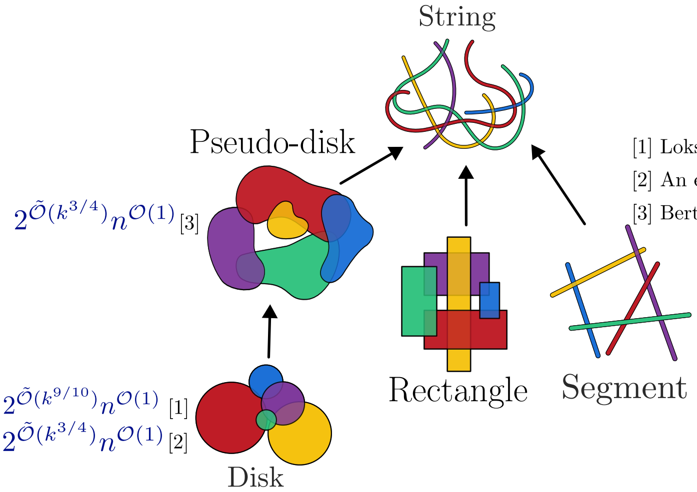

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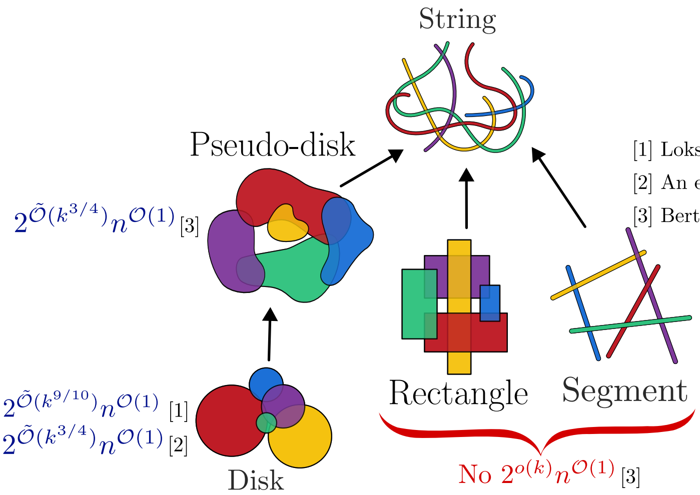

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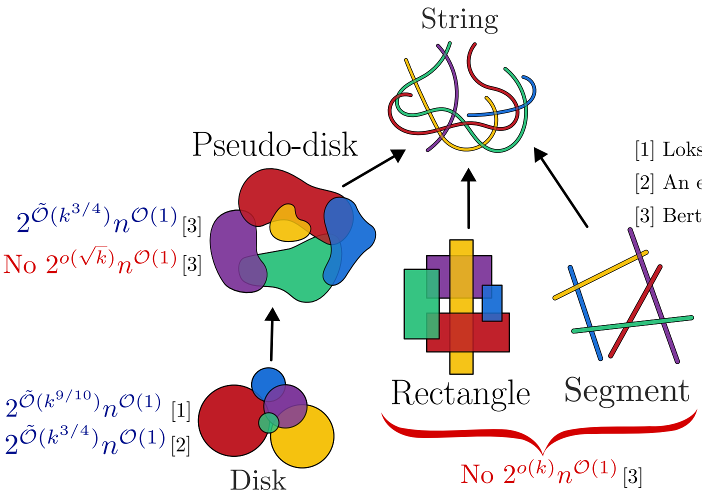

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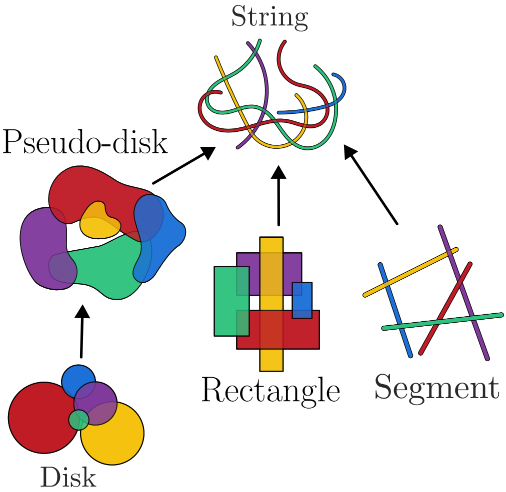
Our results for the K_r-HITTING problem



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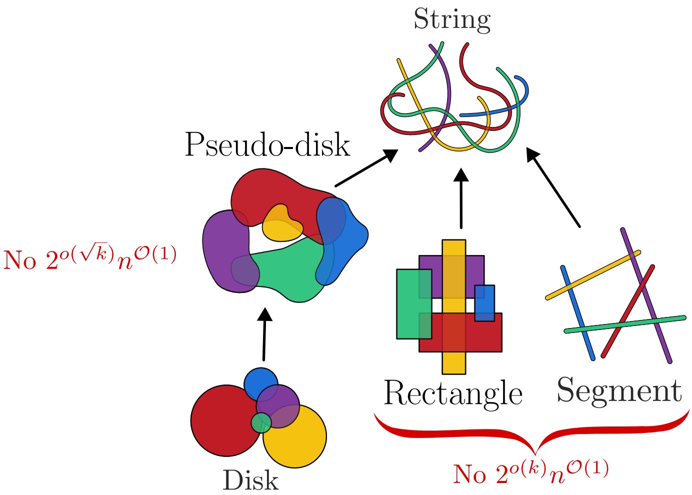

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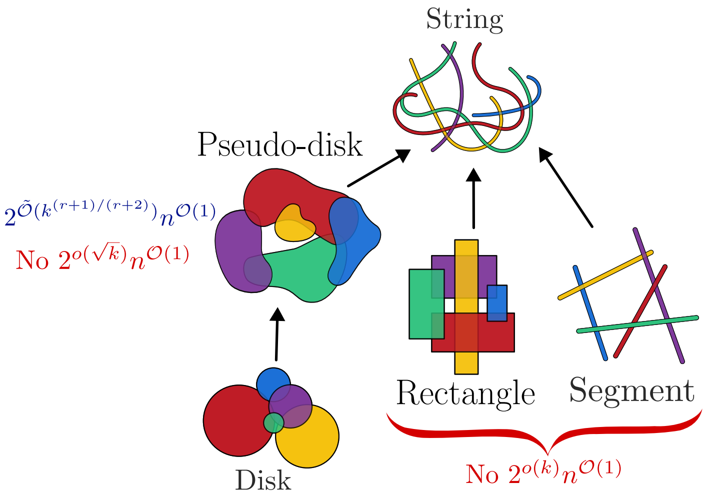

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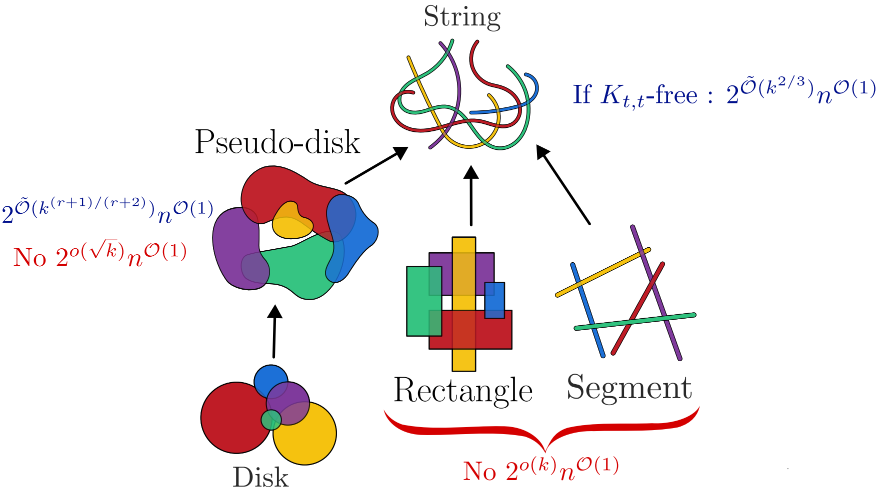

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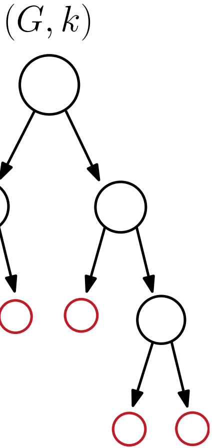

$\mathcal{C}$: instances (G', k')

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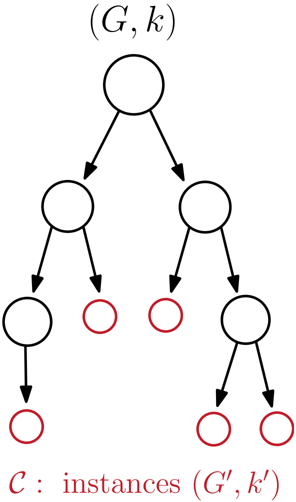

We want for each G' in the instances of $\mathcal{C}$:

- $\omega(G') \leq k^{\epsilon}$ with ϵ to be set later,

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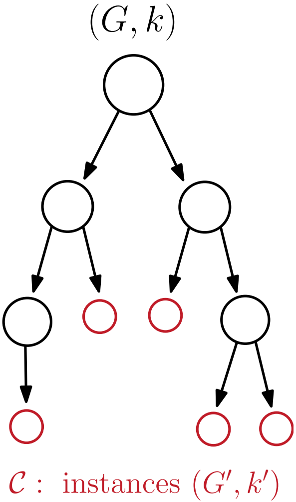

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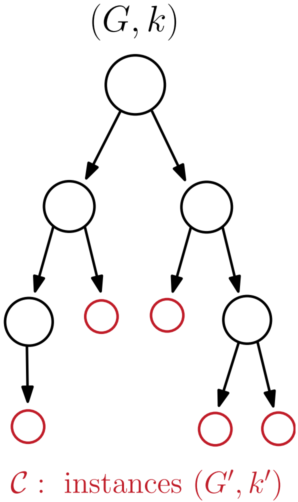

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Then we have for G' pseudo-disk graph :

$$\text{tw}(G') = \sqrt{\omega(G')n(G')} = o(k).$$

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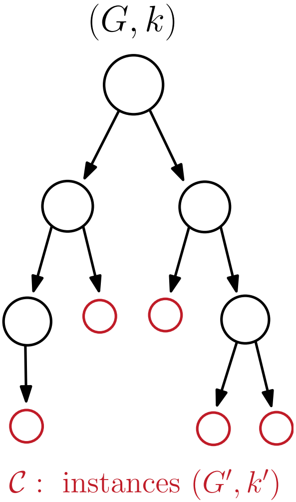

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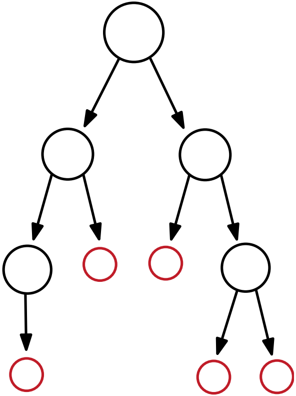

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So if $|\mathcal{C}| = 2^{o(k)}$, the running time is :

$$|\mathcal{C}|2^{\text{tw}(G')}n^{\mathcal{O}(1)} = 2^{o(k)}2^{o(k)}n^{\mathcal{O}(1)}$$
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Preprocessing step : Removing the big cliques

Folklore idea : Branch on cliques for obtaining instances without big cliques.

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Lemma

Given an instance (G, k) of K_r-HITTING and $p = k^{\epsilon}$, we can generate $2^{\tilde{\mathcal{O}}(k/p)}$ instances (G', k') in time $2^{\tilde{\mathcal{O}}(k/p)} n^{\mathcal{O}(1)}$ with :

- G' is K_p-free,
- $k' \leq k$.

Second branching : Example for $r = 3$

We want to obtain instances with $\text{tw} = o(k)$ (or with $n = o(k^2)$ which implies $\text{tw} = o(k)$ in our geometrical graph classes).

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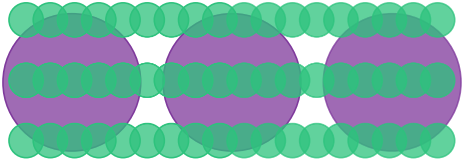
Natural idea :

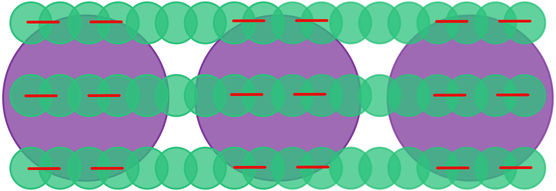
- Assume we find a greedy Triangle Hitting M with $|M| = \mathcal{O}(k)$.
- At this stage the treewidth can still be $\Omega(k)$.



Second branching: ex for $r = 3$

Key idea from Lokshtanov et al. 2022 :

- Source of problem : some $v \in M$ have an edge in their neighborhood outisde M.



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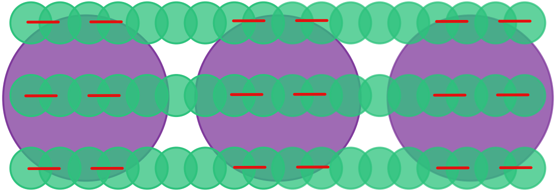
Lokshtanov et al. 2022 propose the following "pseudo branching" routine :

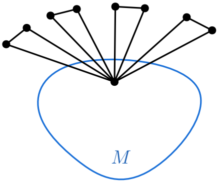
- Input : (G, M, k) where M is a triangle hitting of size $O(k)$.
- Output : a collection (of size $2^{o(k)}$) of (G', M', k') with $M' \supseteq M$ and $|M'| = O(k^{1+\epsilon})$ for $v \in M'$,
- neighborhood outside M' is an independent set.



The pseudo branching routine of Lokshtanov et al. 2022

While there exists $v \in M$ with big ($\geq k^{\epsilon}$) matching in its outside neighborhood :

- Either take v,
- or absord this matching in M (parameter virtually decreases by k^{ϵ}).



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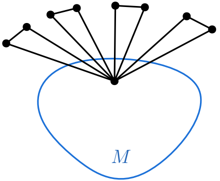
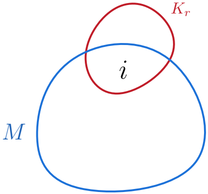


After this, for every $v \in M$, absorb the (small) matching in its neighborhood outside of M.

- in this step: $|M'| = |M|(1 + k^{\epsilon})$.

Back to our contribution for K_r-Hitting

Definition : Given (G, M, k), a type i r-clique X is an r-clique with $|M \cap X| = i$.

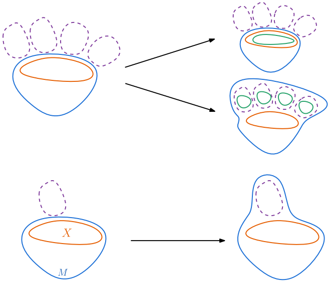


Rethink of what happened for $r = 3$

- After preliminary greedy : No more type 0 3-cliques,
- after virtual branching : No more type 1 3-cliques.

Idea 1 for general r : Given (G, k) , perform r rounds as follows

- After round 0 (greedy): M_0 (no more type 0 r-clique, $|M_0| = O(k)$.
- After round 1: M_1 (no more type 1 r-clique, $|M_1| = O(k^{1+\epsilon})$.
- ...
- After round $r-1$: M_{r-1} (no more type $r-1$ r-clique, $|M_{r-1}| = O(k^{1+r\epsilon})$.



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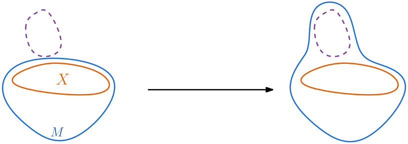
- After round $r-1$: we are done : output M_{r-1}.
- Safe as no r-clique can contain a vertex of $V(G) \setminus M_{r-1}$, kernel = output M_{r-1}

Back to our contribution for K_r-Hitting

Problem : At the end of each round, absorbing the small packing of each "candidate set" X may result in a very large M as :

- We absorb $n_{cand} \times k^{epsilon} r$,
- but n_{cand} could be $|M_{i-1}|^i$.



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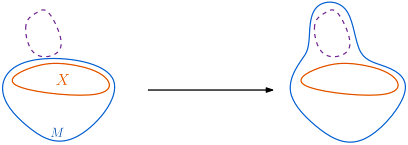

Idea 2 : Challenge this ! Can we really have that many candidates ?

- Each candidate is a clique,
- how many cliques inside M_{i-1} ?

Back to our contribution for K_r-HITTING

Theorem (Chiba et al. 1985)

Any graph G with n vertices and degeneracy d has $\mathcal{O}(id^{i-1}n)$ i-cliques.

Theorem (Berthe et al. 2024)

Pseudo-disk graphs have degeneracy $3e\omega$.

In short : $n_{cand} = \mathcal{O}(\omega^r |M_{i-1}|)$

Obtained results

For a fixed r and $\mu \geq 0$, let $\mathcal{G}$ be a hereditary graph class such that for all $G \in \mathcal{G}$:

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Examples of applications

The K_r-HITTING problem can be solved in time :

- $2^{\tilde{\mathcal{O}}(k^{(r+1)/(r+2)})} n^{\mathcal{O}(1)}$ for **Pseudo-disk graphs** and **map-graphs**.

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- $2^{\tilde{\mathcal{O}}(k^{2/3})} n^{\mathcal{O}(1)}$ for $K_{t,t}$-**free string graphs** and H-**minor-free graphs**.

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3. Conclusion

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Thanks for your attention !