Feedback Vertex Set for pseudo-disk graphs in subexponential FPT time

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Joint work with

Marin Bougeret, Daniel Gonçalves, Jean-Florent Raymond

WG 2024, 19 June 2024

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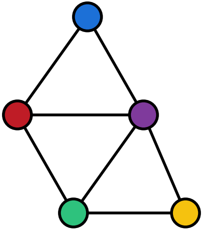
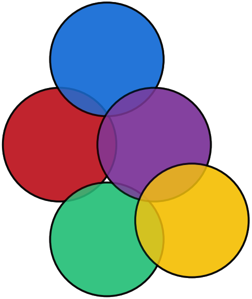
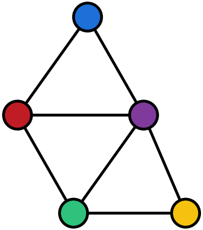
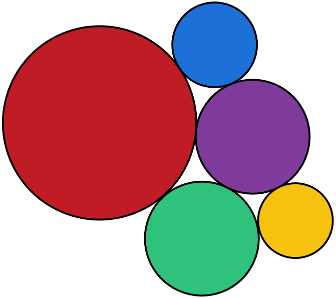
Geometric graphs : Unit disks



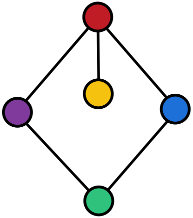


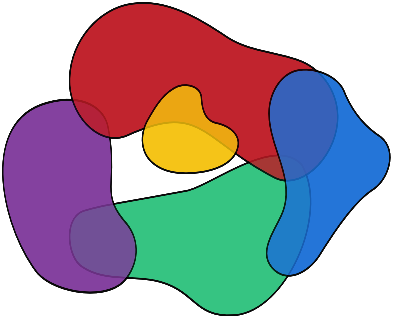
Geometric graphs : Disks in contact = planar graphs



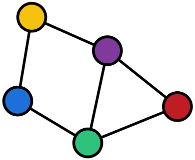


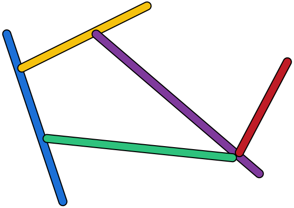
Geometric graphs : Pseudo-disks





Geometric graphs : Segments in contact





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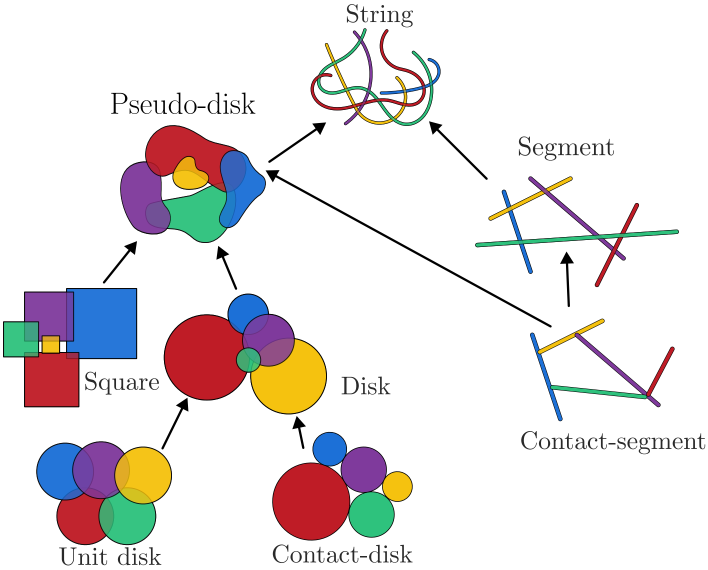
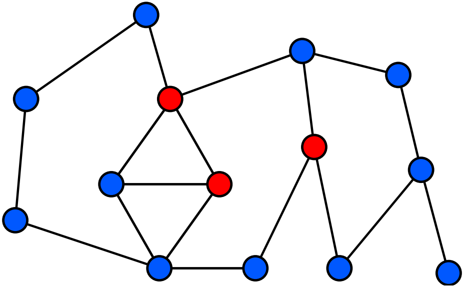


The Feedback Vertex Set problem (FVS)

The Feedback vertex set problem

Can we remove at most k vertices of G such that the resulting graph has no cycle ?



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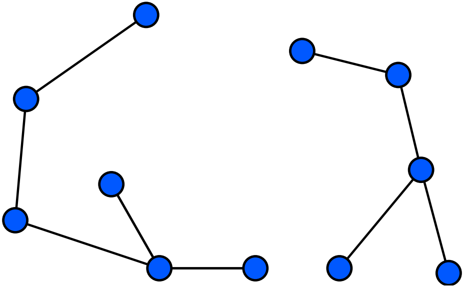

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FVS can be solved in time $2.7^k n^{\mathcal{O}(1)}$.

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What about geometric graph classes ?

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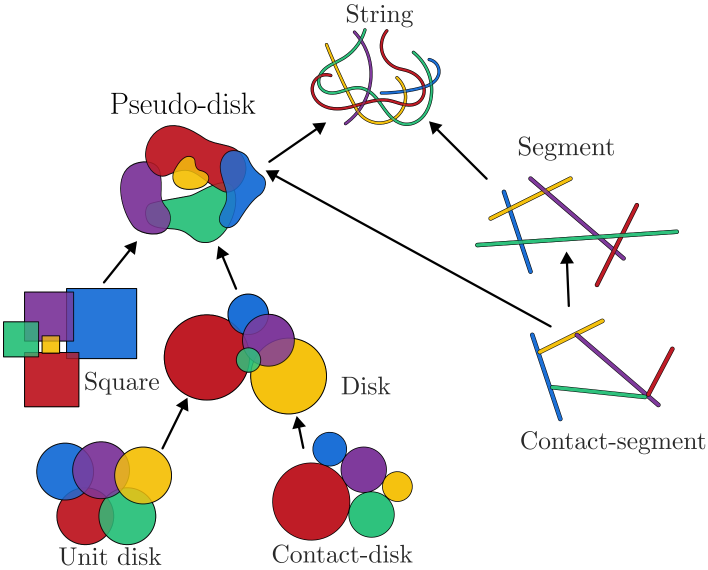

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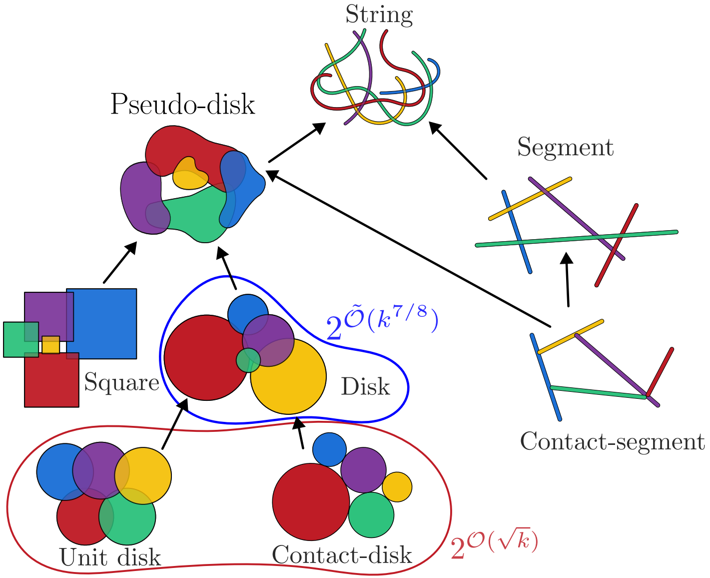

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Definition

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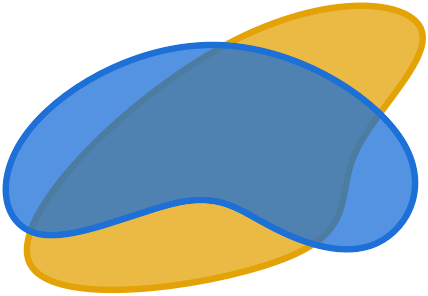

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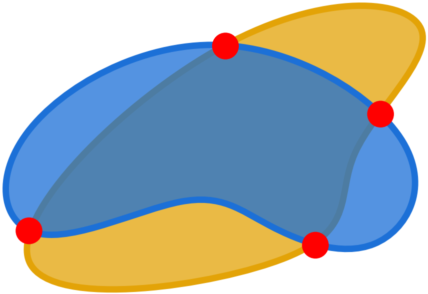
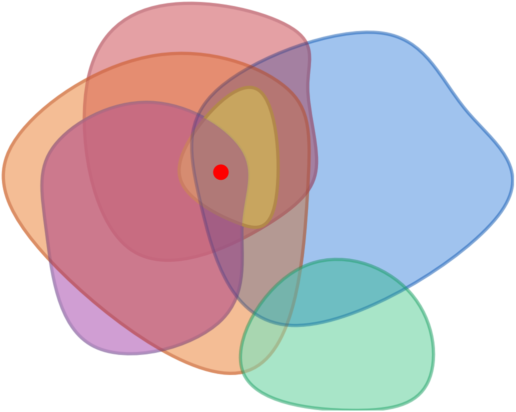


Ply of a set of pseudo-disks



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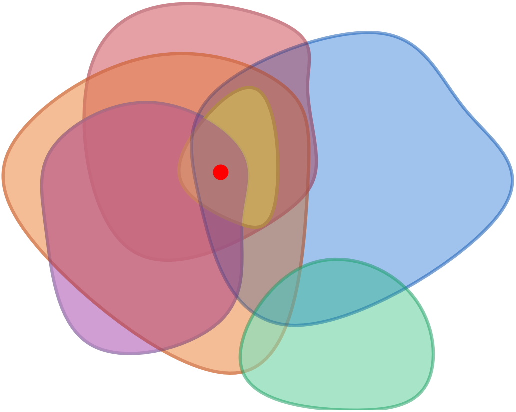
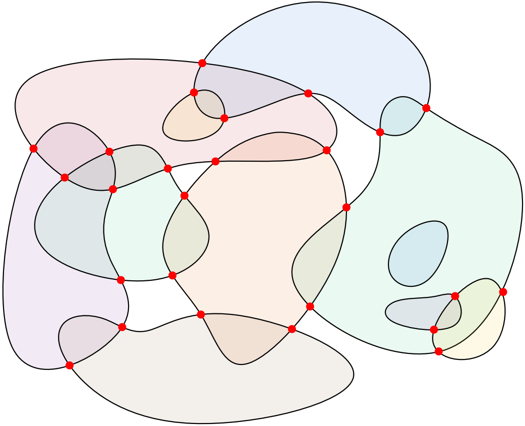


The point is contained in 5 pseudo-disks.
It is maximal : this system of pseudo-disks has ply 5.

Complexity

Theorem (Kedem et al. 1986)

For a set of n pseudo-disks with ply p, the number of regions, arcs and intersection points is $\mathcal{O}(pn)$.



Separators for geometric graphs

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Theorem (Cygan et al. 2011)

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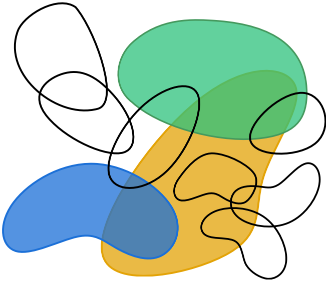
2. Reduce the number of vertices outside M.
3. Bound the treewidth of the obtained kernel.

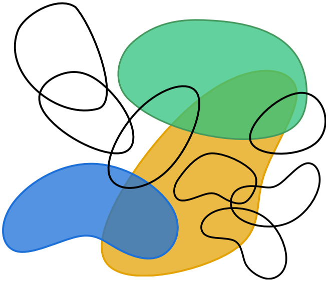
The set M

- M_0: 2-approximation of the solution. $|M_0| = \mathcal{O}(k)$.



The set M

- M_0: 2-approximation of the solution. $|M_0| = \mathcal{O}(k)$.
- M_1: add vertices such that for all $v \in M_1$, $N_G(v) \setminus M_1$ is an independant set (using a branching method from Lokshtanov et al.). $|M_1| = \mathcal{O}(pk)$.



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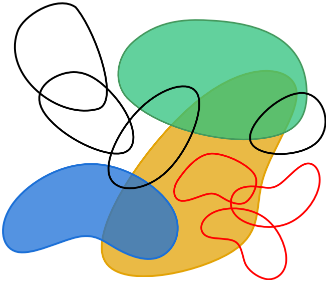

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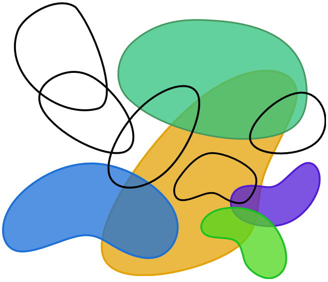

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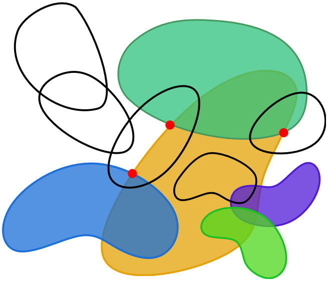

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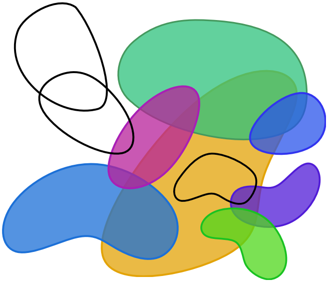
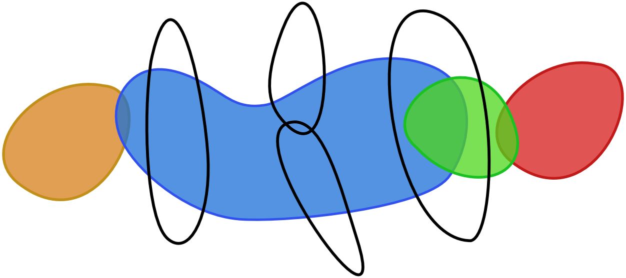


The set M cannot be "crossed"



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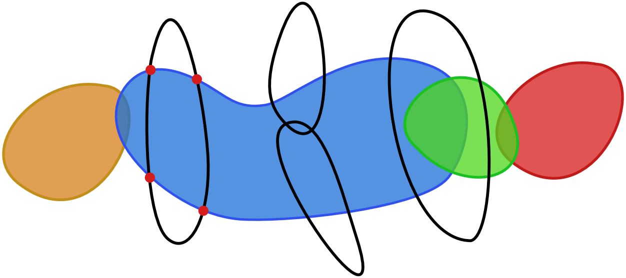
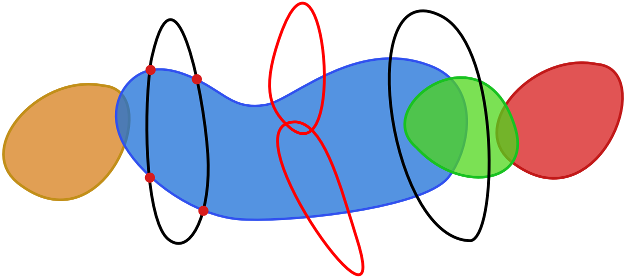


Not a system of pseudo-disks !

The set M cannot be "crossed"



The neighbors of Blue outside M are not independant.

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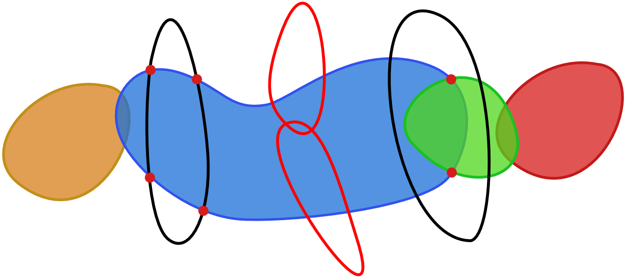
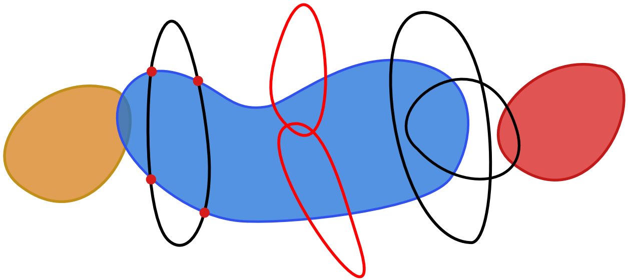


Contains intersection points of pseudo-disks in M.
Either Green or Blue was not in M_1.

The set M cannot be "crossed"



If Blue $\in M_1$ and Green $\notin M_1$,
the neighbors of Blue outside M_1 are not independant.

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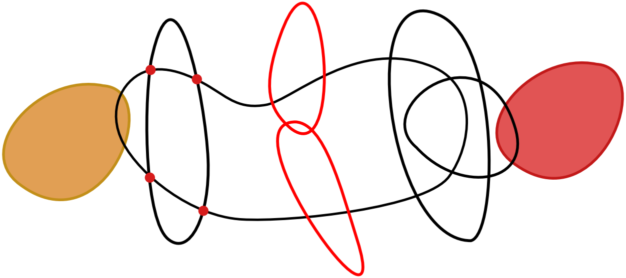
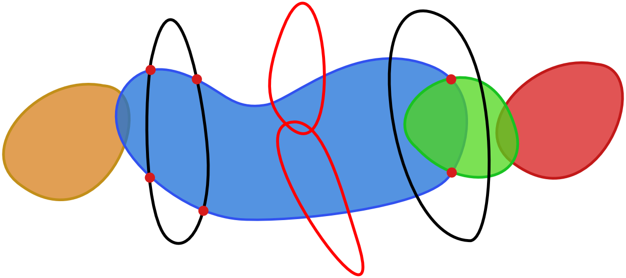


A triangle outside of M_1.

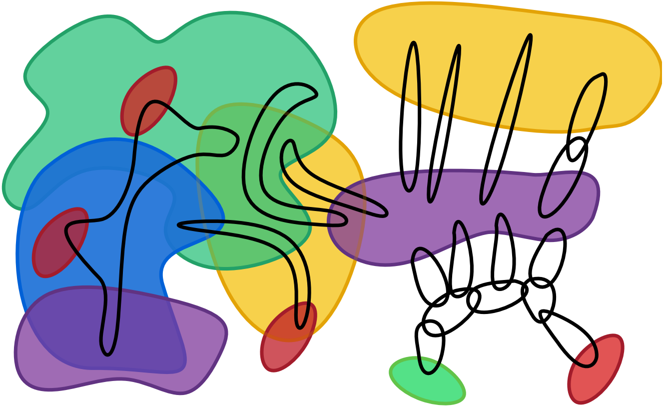
Impossible !

The set M cannot be "crossed"

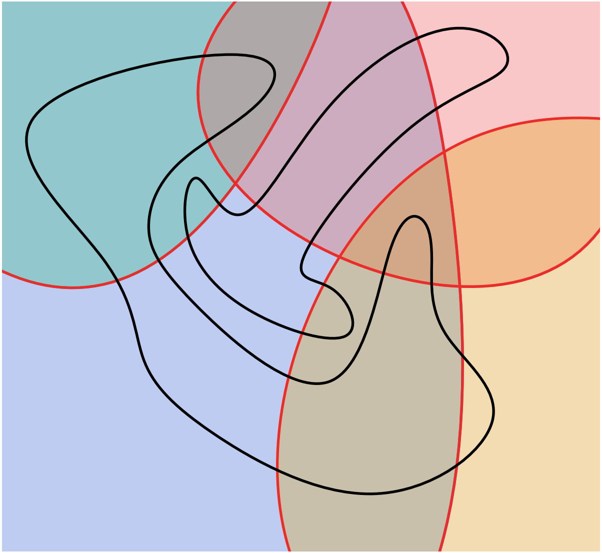


We have no crossing of M,
and the pseudo-disks outside M do not contain intersection points of M.

What remains outside of M



Inner vertices : the case of pseudo-disks with "branching"



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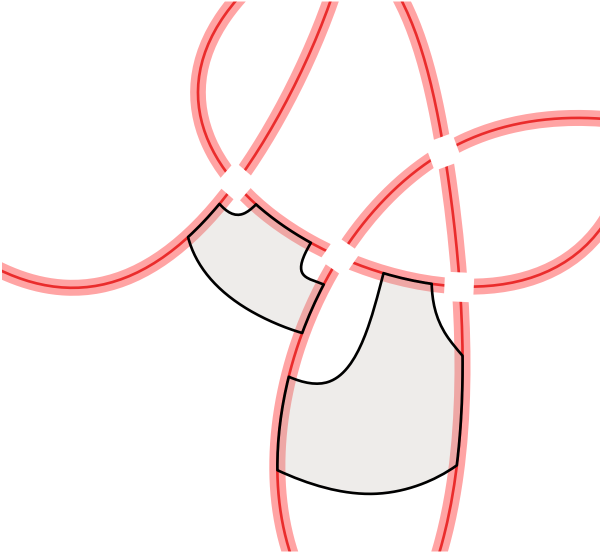

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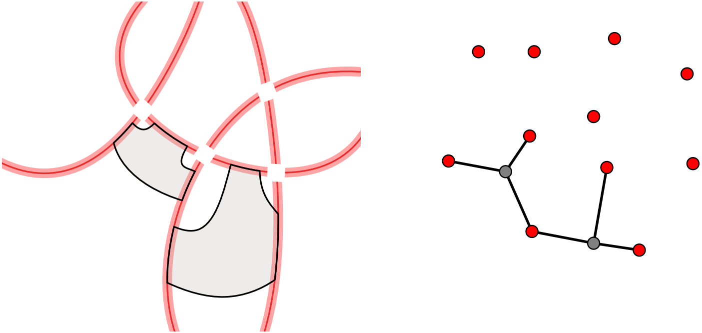

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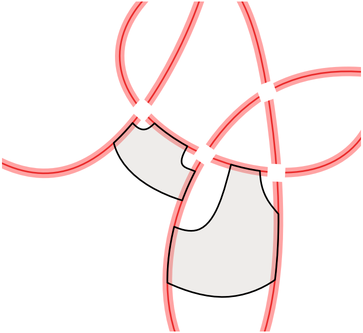
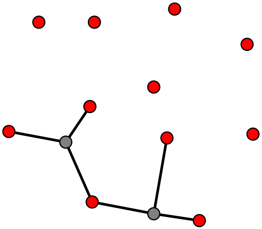
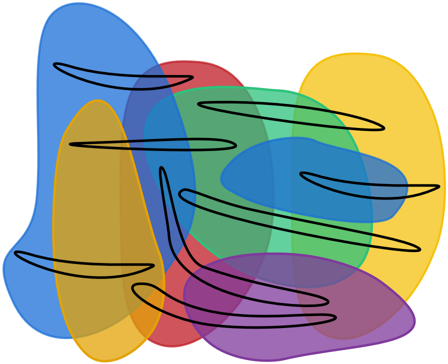




Lemma

For a planar bipartite graph with parts A and B such that for all $v \in A$, $\Delta(v) \geq 3$, we have $|A| = \mathcal{O}(|B|)$.

Inner vertices: the case of paths



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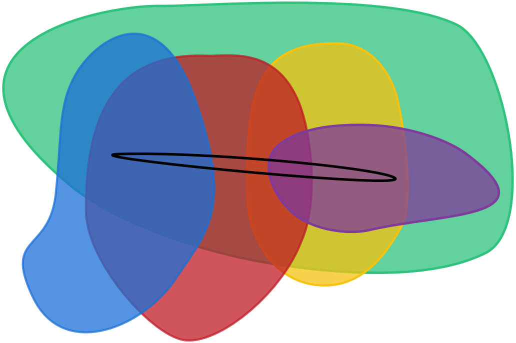

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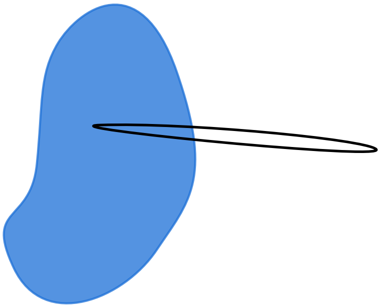

All the neighbors contain at least one endpoint of the path.

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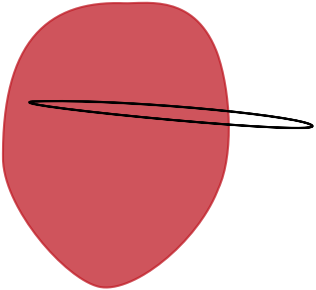

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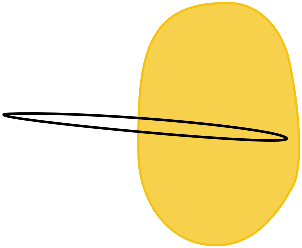

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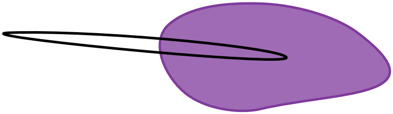

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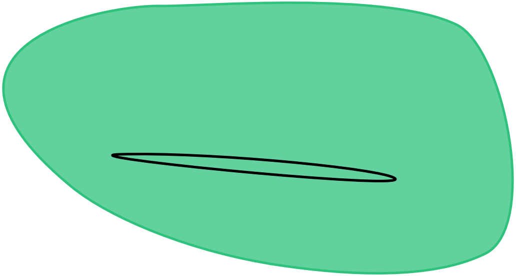

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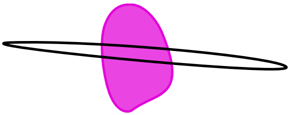
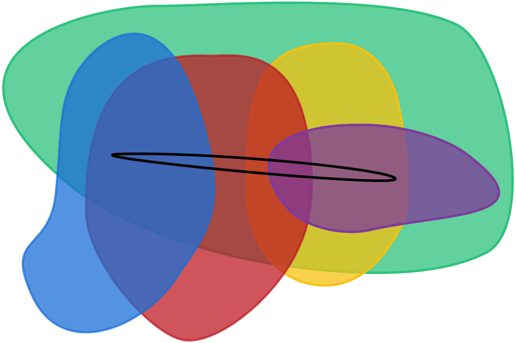


Impossible !

Inner vertices: the case of paths



The neighborhood of a path is the union of the 2 cliques of it extremities.
So it has at most $2p$ neighbors.

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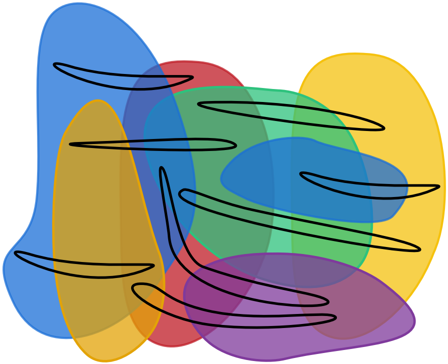

Theorem (Aronov et al. 2020, Keszegh 2018)

Given two sets of pseudo-disks $\mathcal{M}$ and $\mathcal{P}$ where each element of $\mathcal{P}$ intersects at most k elements in $\mathcal{M}$, then the elements of $\mathcal{P}$ have at most $\mathcal{O}(k^3|\mathcal{M}|)$ distinct neighborhoods in $\mathcal{M}$.

A first kernelization rule : removing twins

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Given a set X and $|X| + 2$ independant false twins with neighborhood X, removing one of the twins does not change the size of the optimal FVS.

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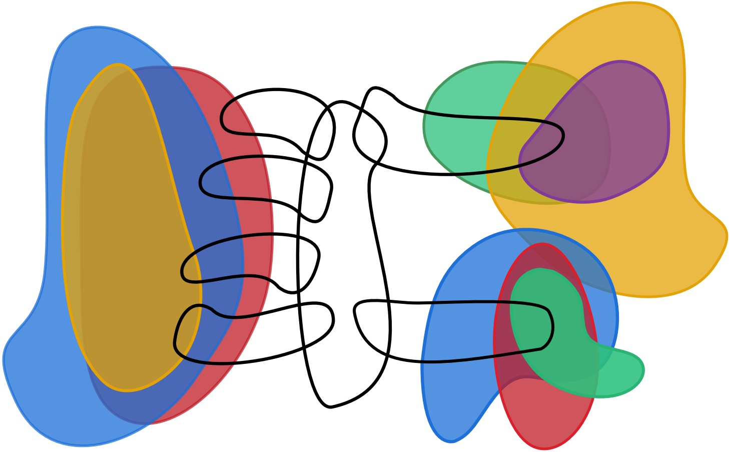
Given a set X and $|X| + 2$ independant false twins with neighborhood X, removing one of the twins does not change the size of the optimal FVS.

Stronger result : In the case where X is the union of two cliques, having 6 twins is enough to ensure that removing a twin is safe.

Remaining trees : chunks



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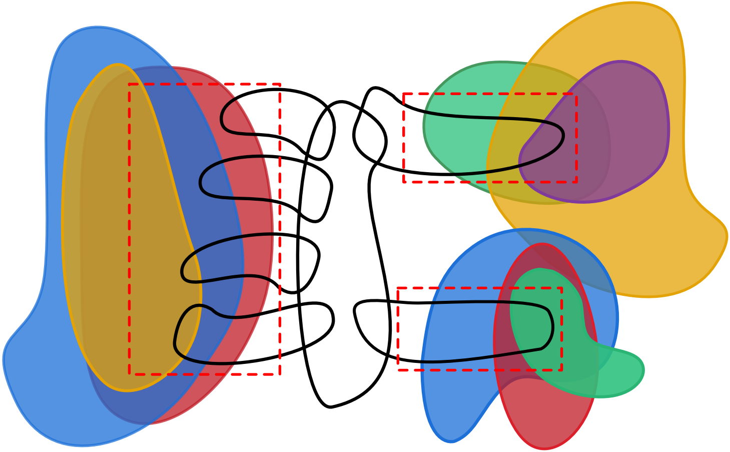

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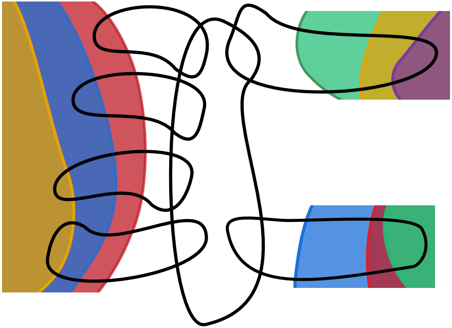

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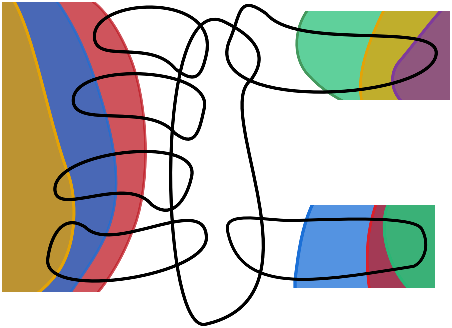

$$\#Chunks(T) = 3.$$

Remaining trees

T a tree of $G - M$.
$\#Chunks(T)$: Number of distinct chunks visited by the tree T.

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- For trees with $\#Chunks(T) = 2$, they can be seen as edges of a planar graph with $p|M|$ vertices.

Remaining trees

T a tree of $G - M$.
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- The number and the size of the trees with $\#Chunks(T) \leq 1$ are bounded by kernelization rules.
- For trees with $\#Chunks(T) = 2$, they can be seen as edges of a planar graph with $p|M|$ vertices.
- For trees with $\#Chunks(T) \geq 3$, we can again define a bipartite planar graph.

The case $\#Chunks(T) \geq 3$

Lemma

$$\sum_{\substack{T \in G-M \\ \#Chunks(T) \geq 3}} \#Chunks(T) = \mathcal{O}(p|M|).$$

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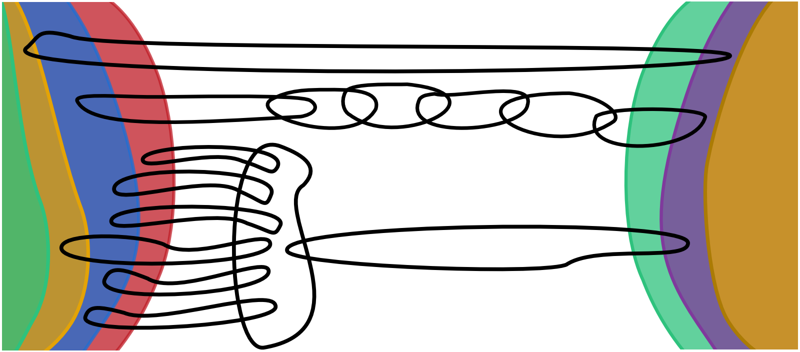
Done by multiple kernelization rules.

Example of kernelization



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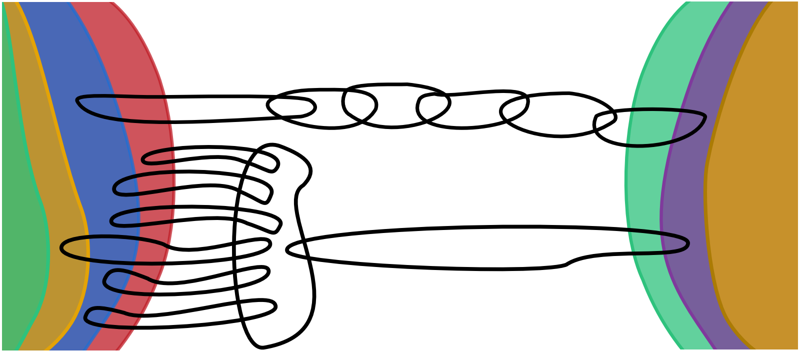

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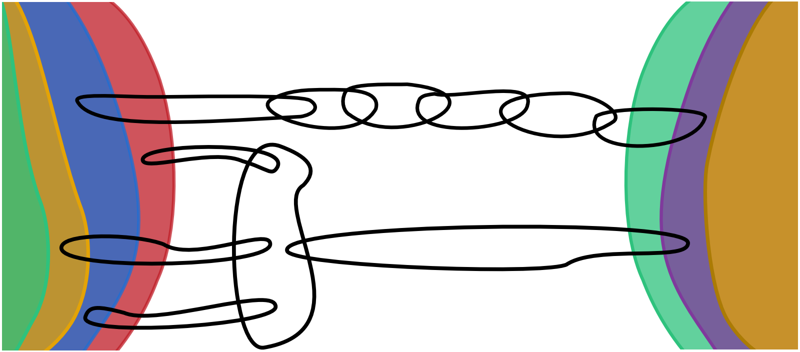

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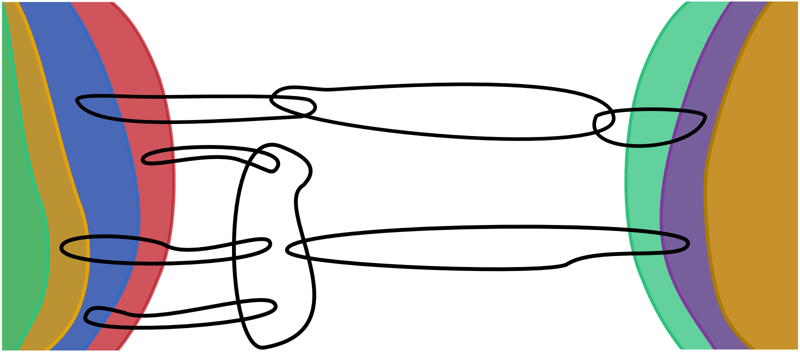

Bounding the treewidth

Lemma

FVS for pseudo-disks of ply p can be reduced to $2^{\tilde{\mathcal{O}}(k^{1-\epsilon})}$ FVS instances (G', k') with $k' \leq k$ and G' pseudo-disk graphs with $\mathcal{O}(kp^4)$ vertices.

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By guessing which vertices of the big cliques are in the solution, we get $p = \tilde{\mathcal{O}}(k^{\epsilon})$ for a price $2^{\tilde{\mathcal{O}}(k^{1-\epsilon})}$.

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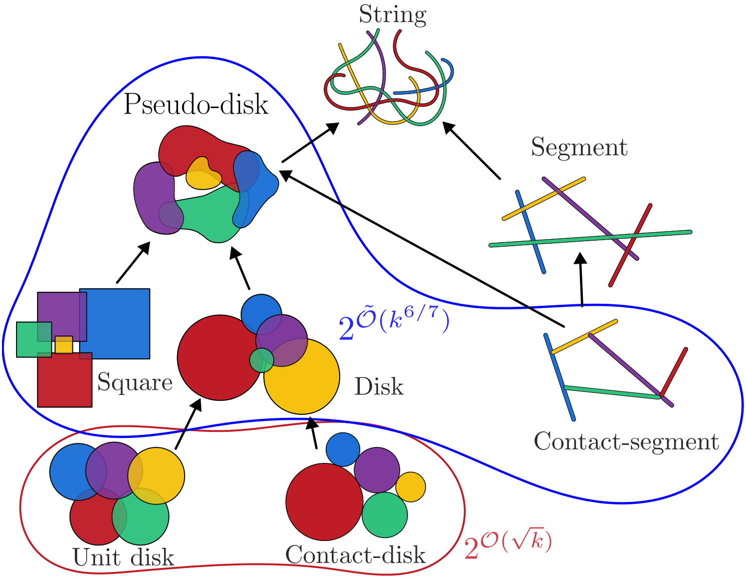
Theorem (this work)

Given a representation, FVS can be solved in time $2^{\tilde{\mathcal{O}}(k^{6/7})} n^{\mathcal{O}(1)}$ in pseudo-disk graphs.

The new cases



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Thanks for your attention !